MATH206A MULTIVARIABLE CALCULUS - PROF. P. WONG

EXAM II - NOVEMBER 3, 2006

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = (yz, xz, xy)$.

(5 pts) (i) Find div F.

For the given F, we have $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$. Since

$$\textbf{div } F = \nabla \cdot F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z},$$

it follows that div $F = 0$.

(5 pts) (ii) Find curl F

$$\text{curl } F = \nabla \times F = (x - x, -(y - y), z - z) = (0, 0, 0).$$

(5 pts) (iii) What is the Jacobian matrix $DF(1, 1, 1)$ of F at $(1, 1, 1)$?

$$DF(1,1,1) = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(5 pts) (iv) Find an approximation of $F(0.9, 1.1, 1.1)$.

The linear approximation asserts that

$$F(\mathbf{x}) = F(\mathbf{a}) + DF(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\mathbf{a} = (1, 1, 1)$. It follows that

$$F(0.9, 1.1, 1.1) \approx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1 \\ 1 \end{bmatrix}.$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (2xy, 3x - y + 5)$.

(10 pts) (i) Suppose that $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a differentiable function such that $g(1, -1, 3) = (2, 5)$ and $Dg(1, -1, 3) = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}$.

What is $D(f \circ g)(1, -1, 3)$?

By the Chain Rule, we have

$$\begin{aligned} D(f \circ g)(1, -1, 3) &= Df(g(1, -1, 3)) \cdot Dg(1, -1, 3) \\ &= Df(2, 5) \cdot \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}. \end{aligned}$$

Since $Df = \begin{bmatrix} 2y & 2x \\ 3 & 1 \end{bmatrix}$**, it follows that** $Df(2, 5) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}$. **Hence,**

$$D(f \circ g)(1, -1, 3) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}.$$

(10 pts) (ii) Suppose that $h : \mathbb{R}^2 \to \mathbb{R}^2$ is another differentiable function such that $Dh(-2, 1) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. What is $D(h \circ f)(-1, 1)$?

Again, by the Chain Rule, we have

$$\begin{aligned} D(h \circ f)(-1, 1) &= Dh(f(-1, 1)) \cdot Df(-1, 1) \\ &= Dh(-2, 1) \cdot Df(-1, 1) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}. \end{aligned}$$

3. Consider the function $f(x, y, z) = x^3y - yz^2 + z^5$.

(8 pts) (i) Find the directional derivative $D_{\mathbf{u}}f(1, 1, 0)$ of f at the point $(1, 1, 0)$ in the direction of $\mathbf{u} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

First, the gradient of f is given by $\nabla f = (3x^2y, x^3 - z^2, -2yz + 5z^4)$ so that $\nabla f(1, 1, 0) = (3, 1, 0)$.

It follows that the directional derivative of f in the direction of u is given by

$$\begin{aligned} D_{\mathbf{u}}f(1,1,0) &= \nabla f(1,1,0) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} \\ &= \frac{(3,1,0)\cdot(1,-1,3)}{\sqrt{11}} \\ &= \frac{2}{\sqrt{11}}. \end{aligned}$$

(12 pts) (ii) Find an equation for the plane tangent to the level surface $f(x, y, z) = 9$ at the point $(3, -1, 2)$.

Since $f(x, y, z) = 9$ is a level surface, we know that $\nabla f(3, -1, 2) = (-27, 23, 84)$ is normal to the plane tangent to the level surface at the point $(3, -1, 2)$.

Thus, this tangent plane is given by the equation

$$(x - 3, y + 1, z - 2) \cdot (-27, 23, 84) = 0.$$

In other words, an equation for this tangent plane is

$$27x - 23y - 84z + 64 = 0.$$

4. Consider the function $f(x, y) = 4y - y^3 - x^2$.

(7 pts) (i) Find all the critical points of f.

The critical points of a smooth function f are precisely the points at which the gradient vanishes. Since $\nabla f = (-2x, 4-3y^2)$, it follows that the critical points are $(0, \frac{2}{\sqrt{3}})$ and $(0, -\frac{2}{\sqrt{3}})$.

(6 pts) (ii) For each of the critical point(s) **a** found in part (i), find the corresponding Hessian matrix $Hf(\mathbf{a})$.

First, the Hessian matrix is given by

$$Hf = \begin{bmatrix} -2 & 0 \\ 0 & -6y \end{bmatrix}.$$

It follows that

$$Hf((0, \frac{2}{\sqrt{3}})) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{12}{\sqrt{3}} \end{bmatrix} \quad \mathbf{a}nd \quad Hf((0, -\frac{2}{\sqrt{3}})) = \begin{bmatrix} -2 & 0 \\ 0 & \frac{12}{\sqrt{3}} \end{bmatrix}.$$

(7 pts) (iii) Use the second derivative test to classify each of the critical point(s) in part (i), i.e., determine whether the critical point is a local max, local min, or saddle point.

At the critical point $(0, \frac{2}{\sqrt{3}})$,

$$\det Hf((0, \frac{2}{\sqrt{3}})) = \frac{24}{\sqrt{3}} > 0 \quad \mathbf{a}nd \quad \frac{\partial^2 f}{\partial x^2} = -2.$$

This implies that $(0, \frac{2}{\sqrt{3}})$ is a local maximum for f. Similarly, at the critical point $(0, -\frac{2}{\sqrt{3}})$,

$$\det Hf((0, -\frac{2}{\sqrt{3}})) = -\frac{24}{\sqrt{3}} < 0.$$

It follows that $(0, -\frac{2}{\sqrt{3}})$ is a saddle point for f.

5. Let C be a smooth path in $\mathbb{R}^3$ given by the parametrization

$$\mathbf{x}(t) = (\ln t, t^2/2, \sqrt{2}t)$$

for $1 \leq t \leq 4$.

(6 pts) (i) Find the length of the path C. [Do your algebra carefully.]

Note that $\mathbf{x}'(t) = (\frac{1}{t}, t, \sqrt{2})$ **so that**

$$\|\mathbf{x}'(t)\| = \sqrt{\frac{1}{t^2} + t^2 + 2} = \sqrt{(\frac{1}{t} + t)^2} = \frac{1}{t} + t.$$

Now, the arc length is given by

$$L = \int_1^4 \|\mathbf{x}'(t)\| \ dt = \int_1^4 \frac{1}{t} + t \ dt = \ln t + \frac{t^2}{2} \Big|_1^4$$
$$= (\ln 4 + 8) - (0 + \frac{1}{2}) = \ln 4 + 8 - \frac{1}{2}.$$

(7 pts.) (ii) Suppose a vertical wall is to be built on top of the path C whose height is given by

$$f(x, y, z) = \frac{e^x y}{z}.$$

Find the surface area of this wall.

The surface area is given by

$$A = \int_1^4 \frac{e^{\ln t} t^2}{2} \frac{1}{\sqrt{2}t} \|\mathbf{x}'(t)\| \ dt = \int_1^4 \frac{e^{\ln t} t^2}{2} \frac{1}{\sqrt{2}t} (\frac{1}{t} + t) \ dt$$
$$= \int_1^4 \frac{t^2}{2\sqrt{2}} (\frac{1}{t} + t) \ dt = \frac{1}{2\sqrt{2}} \int_1^4 t + t^3 \ dt$$
$$= \frac{1}{2\sqrt{2}} (\frac{t^2}{2} + \frac{t^4}{4}) \Big|_1^4 = \frac{1}{2\sqrt{2}} (72 - \frac{3}{4}) = \frac{285}{8\sqrt{2}}.$$

(7 pts) (iii) A force F is acting upon a particle traveling along the path C and $F(x, y, z) = (yz, e^x, z^2)$. What is the total work done $\int_C F \cdot d\mathbf{x}$ of F on the particle?

Note that $d\mathbf{x} = (\frac{1}{t}, t, \sqrt{2}) \ dt$**. It follows that**

$$\int_C F \cdot d\mathbf{x} = \int_1^4 (\frac{t^3\sqrt{2}}{2}, t, 2t^2) \cdot (\frac{1}{t}, t, \sqrt{2}) \ dt$$
$$= \int_1^4 \frac{t^2}{\sqrt{2}} + t^2 + 2\sqrt{2}t^2 \ dt = (\frac{1}{\sqrt{2}} + 1 + \sqrt{2}) \int_1^4 t^2 \ dt$$
$$= (\frac{1}{\sqrt{2}} + 1 + \sqrt{2}) \frac{t^3}{3} \Big|_1^4 = \frac{(3 + \sqrt{2})21}{\sqrt{2}}.$$