INITIALS________

November 3, 2006

1. Find $\displaystyle\int \frac{3x^3-12x^2+17x-3}{x^2-4x+4}\,dx$. Show all your steps.

Since deg(numerator) > deg(denominator), we first carry out the indicated division:

$$\begin{array}{r l} & 3x \quad \leftarrow \text{QUOTIENT} \\ x^2-4x+4 \,\big)\!\!\! & \overline{3x^3-12x^2+17x\;\;-3} \\ & -(3x^3-12x^2+12x) \\ \hline & \qquad\quad 5x-3 \leftarrow \text{REMAINDER} \end{array}$$

So the problem becomes:

$$\int 3x + \frac{5x-3}{x^2-4x+4}\,dx.$$

Since the denominator factors into $(x-2)^2$, we set up partial fractions:

$$\frac{5x-3}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}.$$

Cross-multiplying by $(x-2)^2$ gives: (*)

$$5x-3 = A(x-2) + B \quad \text{[see note @ bottom!]}$$

let $x=2$: $7 = A\cdot 0 + B \Rightarrow B=7$

let $x=0$: $-3 = A(-2) + 7 \Rightarrow A=5$

$$\therefore \int 3x + \frac{5x-3}{(x-2)^2}\,dx =$$

$$\int 3x + \frac{5}{x-2} + \frac{7}{(x-2)^2}\,dx =$$

$$\boxed{\tfrac{3}{2}x^2 + 5\ln|x-2| - 7(x-2)^{-1} + C}$$

(*) note: at this point MANY students "cross-multiplied" incorrectly, winding up with $5x-3 = A(x-2)^2 + B(x-2)$! you get the WRONG A & B values!!!

ALTERNATE UNINTENDED SOL'N:

given $\displaystyle\int \frac{5x-3}{x^2-4x+4}\,dx$, let $u = x^2-4x+4$ so $du = 2x-4\,dx$

$\int$ becomes: $\displaystyle\int \frac{(5/2)(2/5)(5x-3)}{x^2-4x+4}\,dx$

$$= \frac{5}{2}\int \frac{2x-6/5}{x^2-4x+4}\,dx$$

$$= \frac{5}{2}\left[\int \frac{2x-4}{x^2-4x+4}\,dx + \frac{14/5}{(x-2)^2}\,dx\right]$$

$$= \frac{5}{2}\left[\int \frac{du}{u} + \frac{14}{5}\int \frac{1}{(x-2)^2}\,dx\right]$$

$$\vdots\; = \frac{5}{2}\left[\ln|x^2-4x+4| - \frac{14}{5}(x-2)^{-1} + C\right]$$

← Does this look the same?...

$$= \frac{5}{2}\left[\ln|x-2|^2 \quad - \quad \text{''} \quad\right]$$

$$= \frac{5}{2}\left[2\ln|x-2|\right] - \left(\frac{5}{2}\cdot\frac{14}{5}(x-2)^{-1}\right) + C'$$

$$= 5\ln|x-2| - 7(x-2)^{-1} + C'$$

which looks great.

INITIALS_____________

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2. Find $\int \frac{2x+13}{x^2+8x+17}\,dx$. Show all your steps.

There's no division necessary.

The denominator does NOT factor; the integrand is (already) in the form $\left(\frac{Ax+B}{\text{irreducible quadratic}}\right)$

So: let $u = x^2+8x+17$

so $du = 2x+8$

rewrite the integral as

$$\int \frac{2x+8}{x^2+8x+17}\,dx + \int \frac{5}{x^2+8x+17}\,dx$$

which is $\int \frac{du}{u}$,

this integral is $\ln|u| =$

$\ln|x^2+8x+17|$;

Since $x^2+8x+17$ is always positive we may write

$\ln(x^2+8x+17)$

"complete the square": get $\int \frac{5}{x^2+8x+16+1}\,dx$

$$= 5\int \frac{dx}{(x+4)^2+1}$$

let $u = x+4$; then $du = dx$ and the $\int$ is

$$5\int \frac{du}{u^2+1} = 5\arctan u$$

$$= 5\arctan(x+4)$$

Combining these two expressions and adding "+C":

$$\boxed{\ln(x^2+8x+17) + 5\arctan(x+4) + C}$$

3. Find $\int \frac{x^3}{\sqrt{25-x^2}}\,dx$. You may need a trig substitution at the start, and in the middle, it may help to use $\sin^2 u = 1-\cos^2 u$. Show all your steps.

Let $x = 5\sin t$; $dx = 5\cos t\,dt$. [note: you can also do this problem with $x = 5\cos t$]

$\int$ becomes:

$$\int \frac{125(\sin t)^3 \cdot 5\cos t\,dt}{\sqrt{25-25(\sin t)^2}}$$

now, $\sqrt{25-25(\sin t)^2} = 5\sqrt{1-(\sin t)^2} = 5\sqrt{(\cos t)^2} = 5\cos t$ and the integral becomes:

$$\int \frac{125(\sin t)^3 \cdot 5\cos t\,dt}{5\cos t}$$

$= 125\int (\sin t)^3\,dt$

$= 125\int (\sin t)^2(\sin t)\,dt$

$= 125\int 1-(\cos t)^2 \quad \sin t\,dt$

Let $u = \cos t$; $du = -(\sin t)dt$ ...

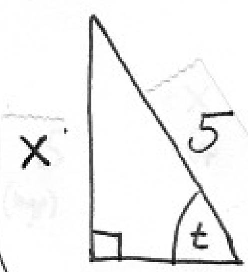
$= -125\int 1-u^2\,du$

$= -125\left(u - u^3/3\right) + C$

$= -125\left(\cos t - \frac{(\cos t)^3}{3}\right) + C$

now we need to "get the x's back"!

Since $x = 5\sin t$, we write $\sin t = \frac{x}{5} = \frac{opp}{hyp}$:



$a = \sqrt{25-x^2}$

(note: $a^2 + x^2 = 5^2$

$a^2 = 5^2 - x^2$

$a = \sqrt{5^2 - x^2}$)

(since $a \geq 0$ no $\pm$ req'd) the fine print

So $\cos t = \frac{adj}{hyp} = \frac{\sqrt{25-x^2}}{5}$

we get $\boxed{-125\left(\frac{\sqrt{25-x^2}}{5} - \left(\frac{\sqrt{25-x^2}}{5}\right)^3 \cdot \frac{1}{3}\right) + C}$

$(\cos t)$

4A. Let $f(x) = \sqrt{x^3} = x^{3/2}$. Find the second-degree Taylor polynomial $P_2(x)$ for f at $x_0 = 16$. Show all your work, neatly organized.

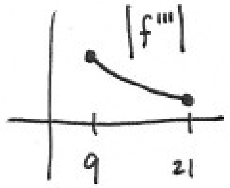
K	$f^{(K)}(x)$	$f^{(K)}(16)$	C_K
0	$x^{3/2}$	$16^{3/2} = 64$	$C_0 = 64$
1	$\frac{3}{2}x^{1/2}$	$\frac{3}{2}16^{1/2} = \frac{3}{2}\cdot 4 = 6$	$C_1 = 6$
2	$\frac{3}{4}x^{-1/2}$	$\frac{3}{4}16^{-1/2} = \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16}$	$C_2 = \frac{1}{2!}\cdot\frac{3}{16} = \frac{3}{32}$
3	$-\frac{3}{8}x^{-3/2}$		

$$\therefore \boxed{P_2(x) = 64 + 6(x-16) + \frac{3}{32}(x-16)^2}$$

4B. What does Taylor's theorem give as the maximum possible error committed by P_2 on the interval [9,21]? (Find your "K_3" to four decimal places)

error $= |P_2(x) - f(x)| < \dfrac{K_3|x-16|^3}{3!}$ where $K_3 = \max$ of $|f'''(x)|$ on $[9, 21]$

by math 105 techniques, or looking at a calculator graph of $|f'''(x)| = \frac{3}{8}\cdot\frac{1}{\sqrt{x^3}}$,

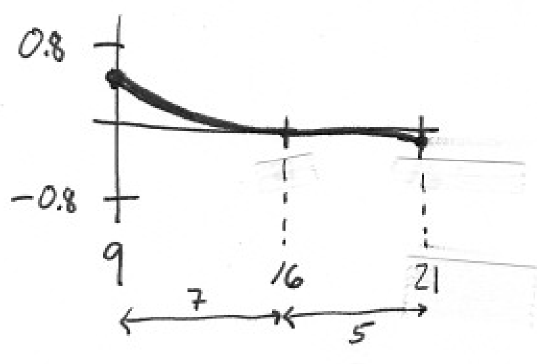


$|f'''(x)|$ is a decreasing function on $[9,21]$ so its max occurs at $x = 9$ and we get $|f'''(9)| = \frac{3}{8}\frac{1}{\sqrt{9^3}} = \frac{3}{8}\frac{1}{27} = \frac{1}{72}$

Take $K_3 = \frac{1}{72} \approx 0.0139$. ALSO, $|x-16|^3$ is "maxed" at $x=9$, and we get $|9-16|^3 = 7^3 = 343$. $\therefore$ max error must be $< \dfrac{0.0139 * 343}{6} \approx 0.7939$

4C. On the interval [9,21], by graphing the difference between $f(x)$ and $P_2(x)$, what does the actual maximimum error appear to be, and at what x does it occur?

Our answer in 4B tells us the graph of $f(x) - P_2(x)$ will fit in a window with $Y_{min} = -0.8$ and $Y_{max} = +0.8$; we see the error is "maxed" @ $x = 9$, and that error is about 0.40625



5A. Show that for $x \geq 3$, $\sqrt[3]{x^6 - 14x} > (1/2)x^2$.

$$\sqrt[3]{x^6 - 14x} > \frac{1}{2}x^2$$

$$\Leftrightarrow \quad x^6 - 14x > \frac{1}{8}x^6 \qquad (a > b \Leftrightarrow a^3 > b^3) \quad (\text{note } (x^2)^3 = x^6 \text{ NOT } x^5)$$

$$\Leftrightarrow \quad \tfrac{7}{8}x^6 - 14x > 0$$

$$\Leftrightarrow \quad \tfrac{7}{8}x^6 > 14x$$

$$\Leftrightarrow \quad x^6 > 16x \qquad (\text{mply by } 8/7)$$

$$\Leftrightarrow \quad x^5 > 16 \qquad (\text{ok. to divide by } x \text{ and keep "}>\text{" b/c } x \text{ is positive})$$

but this true: since $x \geq 3 \Rightarrow x^5 \geq 3^5 = 243$
$\Rightarrow x^5 > 16$ (since $243 > 16$)

5B. Use the inequality in 5A and the appropriate p-test comparison to decide if $\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{x^6 - 14x}}$ converges.

(1) by 5A, $\boxed{0 \leq \dfrac{1}{\sqrt[3]{x^6-14x}} < \dfrac{2}{x^2}}$ ← [NOTE THERE ARE NO $\int$'s in this statement.

IT'S WRONG to START with: "$\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{x^6-14x}} < \int_3^\infty \frac{2}{x^2}dx$"

until you first compare integrands, as in the BOX, & then second, note $\displaystyle\int_3^\infty \frac{2}{x^2}dx$ converges

so third, $\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{x^6-14x}}$ also does and

only NOW it's safe to write $\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{x^6-14x}} < \int_2^\infty \frac{2}{x^2}dx$

(2) by the p test, $\displaystyle\int_3^\infty \frac{2}{x^2}dx$ converges

since $p = 2$ is greater than 1,

(3) and so by the comparison test,

$\displaystyle\int_3^\infty \frac{dx}{\sqrt[3]{x^6-14x}}$ must

converge as well

(and NOW it's safe to write:

"and additionally, $\displaystyle\underbrace{\int_3^\infty \frac{dx}{\sqrt[3]{x^6-14x}} < \int_3^\infty \frac{2}{x^2}dx}$)

BUT N.B. this is NOT your opening line!!

6A. Find $\int \frac{\ln x}{\sqrt{x}}\,dx$. Show your steps.

"LIATE" suggests: Let $u = \ln x$ and $v' = \frac{1}{\sqrt{x}}\,dx = x^{-1/2}\,dx$;

SO $u' = \frac{1}{x}\,dx$ and $v = 2x^{1/2}$

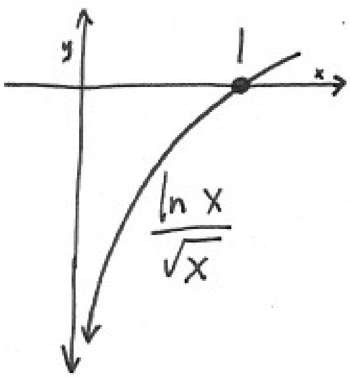
$uv - \int vu'$ becomes:

$$(\ln x)(2x^{1/2}) - \int 2x^{1/2}\cdot\frac{1}{x}\,dx = 2(\ln x)\sqrt{x} - 2\int\frac{dx}{\sqrt{x}}$$

we just DID this integral in finding v above!

$$= 2\ln x\sqrt{x} - 2\cdot 2x^{1/2} + C$$
$$= 2\sqrt{x}\,(\ln x - 2) + C$$

6B. Explain why a graph of $\frac{\ln x}{\sqrt{x}}$ suggests $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ is an improper integral. (Sketch your graph here).



it seems $\frac{\ln x}{\sqrt{x}}$ has a vertical asymptote at $x = 0$.

more specifically, as $a \to 0^+$,

$f(a) \to -\infty$

(or, $f(a)$ decreases without bound as $a \to 0^+$)

6C. Decide if the integral in (6B) converges, and if so to what. Make a table which clearly supports your conclusion. Make sure your work shows "$\lim\limits_{x\to\ldots}$" in all the appropriate places.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(2\sqrt{x}(\ln x - 2)\right)\Big|_a^1$$

a	$-4-2\sqrt{a}(\ln a-2)$
0.1	$-1.2788\ldots$
0.01	$-2.67\ldots$
0.0001	$-3.7757\ldots$
10^{-8}	$-3.9959\ldots$
10^{-20}	$-3.9999\ldots$
↓	↓
0^+	-4

$$= \lim_{a\to 0^+}\left(2\sqrt{1}(\ln 1 - 2) - 2\sqrt{a}(\ln a - 2)\right)$$
$$= \lim_{a\to 0^+}\left(-4 - 2\sqrt{a}(\ln a - 2)\right)$$
$$= -4$$ as the table to the LEFT suggests.

7A. Solve the initial value problem $\begin{cases} \dfrac{dy}{dt} = \dfrac{e^t}{2y+6} \\ y(0) = -5 \end{cases}$.

Note: to solve for y near the end of the problem, try completing the square. Then remember that $A^2 = B$ means $A = \pm\sqrt{B}$.

separate: $2y+6\ dy = e^t\, dt$

Integrate: $\int 2y+6\ dy = \int e^t\, dt$

$y^2 + 6y = e^t + C$

complete the square: $y^2 + 6y + 9 = e^t + 9 + C$ ← (you can replace "9+C" with another constant K) you'll get the right answer. your K will be 3)

$(y+3)^2 = e^t + 9 + C$

$y + 3 = \pm\sqrt{e^t + 9 + C}$ ← $\boxed{\text{SEE NOTE BELOW}}$ (*)

$y = \pm\sqrt{e^t + 9 + C} - 3$

now, $y(0) = -5$. So we need the "−" of the "±" to "get down to" −5:

$-5 = -\sqrt{e^0 + 9 + C} - 3$

$-2 = -\sqrt{10 + C}$

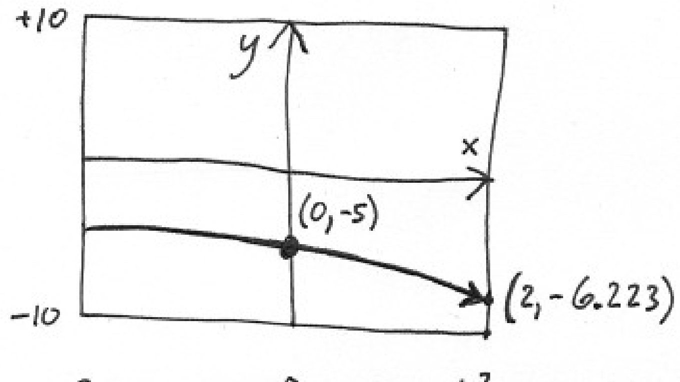
note well the "−" sign!

$10 + C = 4$

$\therefore C = -6$; finally then $y = -\sqrt{e^t + (9-6)} - 3$

i.e. $\boxed{y = -\sqrt{e^t + 3} - 3}$

7B. Graph your solution on the interval $[-2, 2]$. Pick useful y-coordinates.



(*) note you CAN'T take the constant OUT from under the $\sqrt{\ }$ sign; in general $\sqrt{e^t + 9 + C} \neq \sqrt{e^t + 9} + c'$ for any c, c'!

If you disagree, square both sides of $\sqrt{x + A} \stackrel{?}{=} \sqrt{x} + B$ and compare!