Math 106: Review for Exam II

INTEGRATION TIPS

• Substitution: usually let \( w = \) an inside function, especially if \( w' \) is also present in the integrand

• Parts: \( \int u \ dv = uv - \int v \ du \) \quad \text{or} \quad \int u' \ dv = uv - \int u \ v' \ dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPEP will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial”.)

Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x, \arcsin x \))
Algebraic (such as \( x, x^2, x^3 + 4 \))
Trig (such as \( \sin x, \cos 2x \))
Exponentials (such as \( e^x, e^{3x} \))

• Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E\) here) above it on the right.

• Trigonometric Antiderivatives: some useful formulae follow.

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 &= \sec^2 x \\
\cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) &= 2\sin x \cos x
\end{align*}
\]

• Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration

Know the following limits.

\[
\begin{align*}
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to 0^+} e^{-x} &= 0 \\
\lim_{x \to \infty} 1/x &= 0 \\
\lim_{x \to 0^+} 1/x &= \infty \\
\lim_{x \to \infty} \ln x &= \infty \\
\lim_{x \to 0^+} \ln x &= -\infty \\
\lim_{x \to \infty} \arctan x &= \frac{\pi}{2}
\end{align*}
\]
1. Evaluate the following.

Parts
(a) \( \int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} \, dx = \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C \)

Sub
(b) \( \int x^4 e^x \, dx = \int e^w \cdot \frac{1}{5} \, dw = \frac{1}{5} e^w + C = \frac{1}{5} e^{x^5} + C \)

Partial Fractions
(c) \( \int \frac{3x^2 + 2x - 13}{(x-3)(x^2+1)} \, dx = \int \left[ \frac{2}{x-3} + \frac{x+5}{x^2+1} \right] \, dx \)

\( 3x^2 + 2x - 13 = A(x^2+1) + (Bx+C)(x-3) \)

Let \( x = 3: \) \( 20 = A \cdot 10 + 0 \Rightarrow A = 2 \)

Let \( x = 0: \) \( -13 = 2 \cdot 1 + (B \cdot 0 + C)(-3) \)

\( B = 2 - 3C \Rightarrow C = \frac{5}{3} \)

Let \( x = 1: \) \( -8 = 2 \cdot (1^2+1) + (B + 5)(-2) \)

\( B = 1 \)

Long Division
(d) \( \int \frac{4x^3 - 27x^2 + 20x - 17}{x-6} \, dx = \int \left( 4x^2 - 3x + 2 - \frac{5}{x-6} \right) \, dx \)

Improper
(e) \( \int_1^3 \frac{1}{x-1} \, dx = \lim_{t \to 1^+} \int_t^3 \frac{1}{x-1} \, dx = \lim_{t \to 1^+} \ln |x-1| \)

\( = \lim_{t \to 1^+} \left[ \ln |3-1| - \ln |t-1| \right] = \ln 2 - \ln \infty = \infty \Rightarrow \text{integral diverges} \)
2. When you retire from your job, forty-five years from now, you will begin to receive a pension that begins at $100,000 per year and increases at a continuous rate of 3% per year until you die thirty years later. Set up, but do not evaluate, an integral equal to the present value of your pension payments assuming a continuous interest rate of 5%.

\[ PV = \int_{a}^{b} p(t) e^{-rt} \, dt = \int_{45}^{75} 100,000 e^{0.05t} - 0.03t \, dt \]

\[ = \int_{45}^{75} 100,000 e^{-0.02t} \, dt \]
3. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

\[
\begin{align*}
  f(x) &= x^{1/2} \\
  f'(x) &= \frac{1}{2} x^{-1/2} \\
  f''(x) &= -\frac{1}{4} x^{-3/2} \\
  f'''(x) &= \frac{3}{8} x^{-5/2}
\end{align*}
\]

\[
\begin{align*}
  f(100) &= 10 \\
  f'(100) &= \frac{1}{20} \\
  f''(100) &= -\frac{1}{400} \\
  f'''(100) &= -\frac{3}{8000}
\end{align*}
\]

\[
P_2(x) = f(100) + f'(100)(x-100) + \frac{f''(100)}{2!}(x-100)^2
\]

\[
= 10 + \frac{1}{20} (x-100) - \frac{1}{8000} (x-100)^2
\]

4. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval \([100, 110]\)?

\[
|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x-x_0|^{n+1}
\]

\[
|110 - 100|^3 = \frac{3}{8000000}
\]

5. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \[
\int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx
\]

On \([1, \infty)\), \[
\frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}}
\]

So, \[
\int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \geq 5 \int_1^\infty \frac{1}{x^{0.99}} \, dx,
\]

which diverges.

Thus, \[
\int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \text{ must converge}
\]

(b) \[
\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx
\]

On \([1, \infty)\), \[
\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2}
\]

So, \[
\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \leq 4 \int_1^\infty \frac{1}{x^2} \, dx = 4 \lim_{t \to \infty} \int_1^t x^{-2} \, dx
\]

\[
= 4 \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_1^t = 4.
\]

Thus, \[
\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \text{ converges}
\]

and its value is less than 4.