Math 106: Review for Exam II

INTEGRATION TIPS

- Substitution: usually let \( w = \) an inside function, especially if \( w' \) is also present in the integrand.

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below.
(The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial”.)

Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x, \arcsin x \))
Algebraic (such as \( x, x^2, x^3 + 4 \))
Trig (such as \( \sin x, \cos 2x \))
Exponentials (such as \( e^x, e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the numerator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \( (x - 3) \) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \( (Dx + E) \) above it on the right.

- Trigonometric Antiderivatives: some useful formulae follow.

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 &= \sec^2 x \\
\cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) &= 2\sin x \cos x
\end{align*}
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration.

Know the following limits.

\[
\begin{align*}
\lim_{x \to \infty} e^x &= \\
\lim_{x \to -\infty} e^{-x} &= \quad \text{Note: this is the same as } \lim_{x \to -\infty} e^x \\
\lim_{x \to -\infty} 1/x &= \quad \text{Note: the answer is the same for } \lim_{x \to -\infty} 1/x^2 \text{ and similar functions} \\
\lim_{x \to 0^+} 1/x &= \quad \text{Note: the answer is the same for } \lim_{x \to 0^+} 1/x^2 \text{ and similar functions} \\
\lim_{x \to \infty} \ln x &= \\
\lim_{x \to -\infty} \ln x &= \\
\lim_{x \to \infty} \arctan x &=
\end{align*}
\]
1. Evaluate the following.

(a) \( \int x^3 \ln x \, dx \)

(b) \( \int x^4 e^{x^3} \, dx \)

(c) \( \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx \)

(d) \( \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx \)

(e) \( \int_1^3 \frac{1}{x - 1} \, dx \)
2. When you retire from your job, forty-five years from now, you will begin to receive a pension that begins at $100,000 per year and increases at a continuous rate of 3% per year until you die thirty years later. Set up, but do not evaluate, an integral equal to the present value of your pension payments assuming a continuous interest rate of 5%.
3. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

4. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval \([100, 110]\)?

5. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.
   
   (a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

   (b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)