

Math 106: Review for Exam II - SOLUTIONS

INTEGRATION TIPS

- Substitution: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\int u dv = uv - \int v du$ or $\int uv' dx = uv - \int u'v dx$

How to choose which part is u ? Let u be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial".)

Logarithms (such as $\ln x$)

Inverse trig (such as $\arctan x, \arcsin x$)

Algebraic (such as $x, x^2, x^3 + 4$)

Trig (such as $\sin x, \cos 2x$)

Exponentials (such as e^x, e^{3x})

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

| | | | |
|--------------|--------------------|--------------------|--------------------|
| Radical Form | $\sqrt{a^2 - x^2}$ | $\sqrt{a^2 + x^2}$ | $\sqrt{x^2 - a^2}$ |
| Substitution | $x = a \sin t$ | $x = a \tan t$ | $x = a \sec t$ |

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 & \tan^2 x + 1 &= \sec^2 x \\ \sin^2 x &= \frac{1 - \cos(2x)}{2} & \cos^2 x &= \frac{1 + \cos(2x)}{2} \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

| $\int \sin^m x \cos^n x dx$ | Possible Strategy | Identity to Use |
|-----------------------------|---|---|
| m odd | Break off one factor of $\sin x$ and substitute $u = \cos x$. | $\sin^2 x = 1 - \cos^2 x$ |
| n odd | Break off one factor of $\cos x$ and substitute $u = \sin x$. | $\cos^2 x = 1 - \sin^2 x$ |
| m, n even | Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use integration by parts or identities shown to right of this box. | $\sin^2 x = \frac{1 - \cos(2x)}{2}$ $\cos^2 x = \frac{1 + \cos(2x)}{2}$ |

| $\int \tan^m x \sec^n x dx$ | Possible Strategy | Identity to Use |
|-----------------------------|---|---------------------------|
| m odd | Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$. | $\tan^2 x = \sec^2 x - 1$ |
| n even | Break off one factor of $\sec^2 x$ and substitute $u = \tan x$. | $\sec^2 x = \tan^2 x + 1$ |
| m even, n odd | Use identity at right to reduce to powers of $\sec x$ alone. Then use integration by parts or reduction formula (if allowed). | $\tan^2 x = \sec^2 x - 1$ |

Useful Trigonometric Derivatives and Antiderivatives

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

- Improper integrals: look for ∞ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow \infty} e^{-x} = 0$$

Note: this is the same as $\lim_{x \rightarrow -\infty} e^x$.

$$\lim_{x \rightarrow \infty} 1/x = 0$$

Note: the answer is the same for $\lim_{x \rightarrow \infty} 1/x^2$ and similar functions.

$$\lim_{x \rightarrow 0^+} 1/x = \infty$$

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$$\lim_{x \rightarrow \infty} \ln x = \infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$$\lim_{x \rightarrow \infty} \arctan x = \pi/2$$

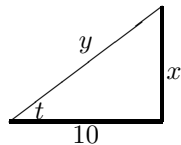
1. Evaluate the following.

- (a) Let $u = \sin x$, so $du = \cos x dx$.

$$\begin{aligned} \int \sin^6 x \cos^3 x dx &= \int \sin^6 x (1 - \sin^2 x) \cos x dx \\ &= \int u^6 (1 - u^2) du \\ &= \int (u^6 - u^8) du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + C \\ &= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C \end{aligned}$$

Use $\cos^2 x = 1 - \sin^2 x$.

- (b) Let $x = 10 \tan t$, so $dx = 10 \sec^2 t dt$.



$$x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100}$$

$$\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10}$$

$$\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{100+x^2}} &= \int \frac{10 \sec^2 t \, dt}{\sqrt{100+100 \tan^2 t}} \\
&= \int \frac{10 \sec^2 t \, dt}{10\sqrt{1+\tan^2 t}} && \text{Now use } 1 + \tan^2 t = \sec^2 t. \\
&= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}} \\
&= \int \sec t \, dt \\
&= \ln |\sec t + \tan t| + C && \text{Now use triangle above.} \\
&= \ln \left| \frac{\sqrt{x^2+100}}{10} + \frac{x}{10} \right| + C
\end{aligned}$$

(c) This is an improper integral, so we need to use a limit.

$$\begin{aligned}
\int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx &= \lim_{t \rightarrow \infty} \int_3^t \frac{1}{x(\ln x)^{100}} \, dx \\
&= \lim_{t \rightarrow \infty} \int_{x=3}^{x=t} \frac{1}{u^{100}} \, du && \text{Substitute } u = \ln x, \text{ so } du = \frac{dx}{x}. \\
&= \lim_{t \rightarrow \infty} \frac{u^{-99}}{-99} \Big|_{x=3}^{x=t} \\
&= \lim_{t \rightarrow \infty} \frac{-1}{99(\ln x)^{99}} \Big|_3^t \\
&= \lim_{t \rightarrow \infty} \left[\frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \\
&= 0 - \frac{-1}{99(\ln 3)^{99}} \\
&= \frac{1}{99(\ln 3)^{99}} && \text{So, the integral converges (to this value).}
\end{aligned}$$

(d) We'll use integration by parts: $u = x \Rightarrow du = dx$ and $dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}$.

$$\begin{aligned}
\int_0^\infty x e^{-2x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-2x} \, dx \\
&= \lim_{t \rightarrow \infty} \left[x \frac{e^{-2x}}{-2} \Big|_0^t - \int_0^t \frac{e^{-2x}}{-2} \, dx \right] \\
&= \lim_{t \rightarrow \infty} \left[x \frac{e^{-2x}}{-2} - \frac{e^{-2x}}{4} \right]_0^t \\
&= \lim_{t \rightarrow \infty} \left[\frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} \right]_0^t \\
&= \lim_{t \rightarrow \infty} \left[\frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}} \right] - \left[\frac{0}{2e^0} - \frac{1}{4e^0} \right] \\
&= (0 - 0) - (0 - 1/4) \\
&= 1/4 && \text{So, the integral converges (to this value).}
\end{aligned}$$

(e) Partial Fractions:

Write $\frac{3x^2 + 2x - 13}{(x-3)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$. Now multiply both sides by $(x-3)(x^2+1)$ to get

$$3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).$$

Let $x = 3$. Then $20 = C(10)$, so $C = 2$.

Let $x = 0$. Then $-13 = B(-3) + 2(1)$, so $B = 5$.

Let $x = 1$. Then $-8 = (A(1) + 5)(-2) + 2(2)$, so $A = 1$.

$$\begin{aligned} \int \frac{3x^2 + 2x - 13}{(x-3)(x^2+1)} dx &= \int \left[\frac{x+5}{x^2+1} + \frac{2}{x-3} \right] dx \\ &= \int \left[\frac{x}{x^2+1} + \frac{5}{x^2+1} + \frac{2}{x-3} \right] dx && \text{Let } u = x^2 + 1, \text{ so } du = 2x dx. \\ &= \int \frac{\frac{1}{2} du}{u} + \int \left[\frac{5}{x^2+1} + \frac{2}{x-3} \right] dx \\ &= \frac{\ln u}{2} + 5 \arctan x + 2 \ln |x-3| + D \\ &= \frac{\ln(x^2+1)}{2} + 5 \arctan x + 2 \ln |x-3| + D \end{aligned}$$

- (f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

$$\begin{array}{r} 4x^2 - 3x + 2 + \frac{-5}{x-6} \\ x-6 \overline{) 4x^3 - 27x^2 + 20x - 17} \\ \underline{4x^3 - 24x^2} \\ -3x^2 \\ \underline{-3x^2 + 18x} \\ 2x \\ \underline{2x - 12} \\ -5 \end{array}$$

Now, we compute the integral.

$$\int \frac{4x^3 - 27x^2 + 20x - 17}{x-6} dx = \int \left[4x^2 - 3x + 2 - \frac{5}{x-6} \right] dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x-6| + C$$

- (g) This integral is improper at $x = 1$ because the integrand has a vertical asymptote there.

$$\begin{aligned} \int_1^3 \frac{1}{x-1} dx &= \lim_{t \rightarrow 1^+} \int_t^3 \frac{1}{x-1} dx \\ &= \lim_{t \rightarrow 1^+} \ln |x-1| \Big|_t^3 \\ &= \lim_{t \rightarrow 1^+} [\ln |3-1| - \ln |t-1|] \end{aligned}$$

Since $\lim_{t \rightarrow 1^+} (-\ln |t-1|) = \infty$, this integral diverges (to ∞).

2. Find the second-degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 100$.

$$\begin{array}{ll} f(x) = x^{1/2} & f(100) = 10 \\ f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} & f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} \\ f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4x^{3/2}} & f''(100) = \frac{-1}{4 \cdot 100^{3/2}} = \frac{-1}{4000} \end{array}$$

$$\begin{aligned}
P_2(x) &= f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2 \\
&= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
\end{aligned}$$

3. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval $[100, 110]$?

We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 100$, and $x = 110$ (the farthest from x_0 that we are considering).

$$K_3 = \max \text{ of } |f'''(x)| \text{ on } [100, 110] = \max \text{ of } \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 110] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}$$

Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{\frac{3}{800,000}}{3!}|110 - 100|^3 = \frac{1}{1600}$.

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) $\int_1^\infty \frac{6 + \cos x}{x^{0.99}} dx$

For all $x \geq 1$, we have $\frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}}$ because the minimum value of $\cos x$ is -1 .

Since $\int_1^\infty \frac{5}{x^{0.99}} dx$ diverges (compute yourself or notice that $p = 0.99 < 1$), we know that the integral in question must diverge too.

(b) $\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} dx$

For all $x \geq 1$, we have $\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = 4\frac{1}{x^2}$. (We've made the denominator smaller and the numerator larger, so the new fraction is larger.)

$$\begin{aligned}
4 \int_1^\infty \frac{dx}{x^2} &= 4 \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} \\
&= 4 \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_1^t \\
&= 4 \lim_{t \rightarrow \infty} \left[\frac{-1}{t} - \frac{-1}{1} \right] \\
&= 4[0 - (-1)] \\
&= 4
\end{aligned}$$

Therefore, the original integral in question must converge to a value less than 4.

5. The probability density function (pdf) of the length (in minutes) of phone calls on a certain wireless network is given by $f(x) = ke^{-0.2x}$ where x is the number of minutes. Note that the domain is $x \geq 0$ since we can't have a negative number of minutes.

- (a) What must be the value of k ?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$\begin{aligned}
 \int_0^{\infty} ke^{-0.2x} dx &= \lim_{t \rightarrow \infty} \int_0^t ke^{-0.2x} dx \\
 &= \lim_{t \rightarrow \infty} \left. \frac{ke^{-0.2x}}{-0.2} \right|_0^t \\
 &= \lim_{t \rightarrow \infty} \frac{ke^{-0.2t}}{-0.2} - \frac{ke^0}{-0.2} \\
 &= 0 - \frac{k}{-0.2} \\
 &= 5k
 \end{aligned}$$

So, we have $5k = 1$ or $k = 0.2$.

(b) **What fraction of calls last more than 3 minutes?**

$$\begin{aligned}
 \int_3^{\infty} 0.2e^{-0.2x} dx &= \lim_{t \rightarrow \infty} \int_3^t 0.2e^{-0.2x} dx \\
 &= \lim_{t \rightarrow \infty} \left. \frac{0.2e^{-0.2x}}{-0.2} \right|_3^t \\
 &= \lim_{t \rightarrow \infty} -e^{-0.2t} - (-e^{-0.6}) \\
 &= 0 + e^{-0.6} \\
 &= e^{-0.6} \approx 0.5488
 \end{aligned}$$

Note that we could instead have computed $1 - \int_0^3 0.2e^{-0.2x} dx$ and gotten the same answer, but the point of introducing pdf's in this text seems to be to show how improper integrals are used.