TEST 1

Math 205
10/10/11

Name: $\underbrace{\text{Key :)}}_{\text{by writing my name i swear by the honor code}}$

Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can be reduced into RREF with calculator without showing steps). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements. Put a smiley face next to your name for one point.
- This test has 8 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*13 points*) Consider the system of equations:

$$\begin{array}{rcrcrcrcrcr} x & + & 6y & + & 2z & - & 5u & - & 2v & = & -4 \\ & & & & 2z & - & 8u & - & v & = & 3 \\ & & & & & & & & v & = & 7 \end{array}$$

a. (*5 pts*) Write the system as a matrix equation.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$$

b. (*8 pts*) Solve the system and use vector parameter form for your solution.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \quad y, u \text{ free}$$

$x = -6y - 3u$

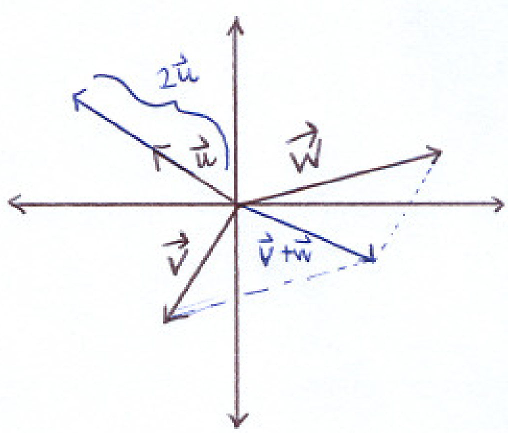
$z = 5 + 4u$

$v = 7$

$$\begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + s \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \quad \text{for all } s, t \in \mathbb{R}$$

2. (*10 points*) Consider the vectors $\vec{u}, \vec{v}, \vec{w}$ as labelled on the graph.

a. (*5 pts*) Sketch and label $2\vec{u}$ and $\vec{v} + \vec{w}$.



b. (*5 pts*) Describe, geometrically and in words, the space Span$\{\vec{u}, \vec{v}, \vec{w}\}$.

Span$\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^2$, plane

all linear combinations of $\{\vec{u}, \vec{v}, \vec{w}\}$

Span$\{\vec{u}, \vec{v}, \vec{w}\} = \{c_1\vec{u} + c_2\vec{v} + c_3\vec{w} \mid c_1, c_2, c_3 \in \mathbb{R}\}$

3. (*20 points*)

a. (*3 pts*) Give the definition of what it means for the set of vectors $S = \{\vec{x_1}, \vec{x_2}, \ldots \vec{x_p}\}$ to be linearly independent. (Don't tell me the method you would use to SHOW they are linear independent. I am looking for the definition.)

If $c_1\vec{x_1} + c_2\vec{x_2} + \ldots + c_p\vec{x_p} = \vec{0}$ has only the trivial solution, $c_1 = c_2 = c_3 = \ldots = c_p = 0$ then S is a set of linearly independent vectors.

Now, suppose $S = \{\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}\}$ be the columns vectors of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & -1 & 3 \\ 1 & -2 & 1 & 1 \\ -2 & -1 & 1 & -3 \end{bmatrix}$

b. (*5 pts*) Find all solutions of $A\vec{x} = \vec{0}$. Hint: $RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ let $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$

C_4 is free

$c_1 = -4/3$

$c_2 = 0$

$c_3 = 1/3\, c_4$

c_4 free

OR $c_4 \begin{bmatrix} -4/3 \\ 0 \\ 1/3 \\ 1 \end{bmatrix}$ for any $c_4 \in \mathbb{R}$.

c. (*4 pts*) Express $\vec{0}$ as a linear combination of the columns of A where the scalar associated with $\vec{x_3}$ is one or explain why this is impossible.

From b, let $c_4 = 3$ then $\vec{x} = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ solves $A\vec{x} = \vec{0}$.

So, $-4\vec{x_1} + \vec{x_3} + 3\vec{x_4} = \vec{0}$

d. (*5 pts*) Express $\vec{x_3}$ as a linear combination of the other columns or explain why this is impossible.

From c,

$\vec{x_3} = 4\vec{x_1} - 3\vec{x_4}$

e. (*3 pts*) Given some vector $b \in \mathbb{R}^4$, can you always solve $A\vec{x} = \vec{b}$? Explain.

By Thm 4., since A doesn't have a pivot in every row we <u>cannot</u> always solve $A\vec{x} = \vec{b}$ for any $\vec{b}$.

4. (*12 points*) One way to see if a linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one is to check which vectors $\vec{x}$ in the domain are mapped to the zero vector, $\vec{0}$ (ie. Find all $\vec{x}$ such that $T(\vec{x}) = A\vec{x}) = \vec{0}$.) We know $\vec{x} = \vec{0}$ will work, but there may be more in which case T is not one-to-one.

a. (*6 pts*) Find an example of a non-trivial (not the zero matrix) linear transformation that maps a non-zero vector to the zero vector. Give a matrix, A, and a vector, $\vec{x}$, and show how A maps your chosen $\vec{x}$ to the zero vector.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b. (*6 pts*) A linear transformation T is one-to-one if and only if the ONLY vector that T maps to $\vec{0}$ is $\vec{x} = \vec{0}$ (ie. only the trivial solution). Explain why if $T(\vec{x}) = 0$ has a unique solution, $\vec{x} = \vec{0}$, then T is one-to-one.

If $T(\vec{x}) = 0$ then $A\vec{x} = \vec{0}$. If $\vec{x} = 0$ is the only solution then A has a pivot in every column and no free variables. However, if A has a pivot in every column then $T(\vec{x}) = A\vec{x}$ is one-to-one.

5. (*15 points*) T is a linear transformation. $T : \mathbb{R}^2 \to \mathbb{R}^3$.

a. (*8 pts*) Can T be one-to-one? Can T be onto? Explain.

$T(\vec{x}) = A\vec{x}$ $\qquad A = \begin{array}{c c} & 2 \\ 3 & \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \end{array}$

T can be one-to-one if there is a pivot in every column of A.

T can never be onto because you will never have a pivot in every row.

b. (*7 pts*) $T(\vec{e_1}) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $T(\begin{bmatrix} -1 \\ 1 \end{bmatrix}) = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. Find $T(\begin{bmatrix} 3 \\ 2 \end{bmatrix})$.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Since T is a linear transformation, $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = 5T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 5a + 2d \\ 5b + 2e \\ 5c + 2f \end{bmatrix}$$

6. (*10 points*) True or False. No explanation necessary.

- T Every elementary row operation is invertible.
- T When $\vec{u}$ and $\vec{v}$ are nonzero vectors, Span$\{\vec{u}, \vec{v}\}$ contains the line through $\vec{u}$ and the origin.
- F Whenever a system has free variables, the solutions set contains many solutions.
- F The columns of a 4 x 2 matrix always span $\mathbb{R}^2$.
- T My favorite theorem used to be Theorem 4, but now it is the Invertible Matrix Theorem.

7. (*10 points*) Given A is an $m \times n$ matrix and B is an $r \times t$ matrix.

a. (*2 pts*) Explain what must be true of m, n, r and t for AB to exist.

$n = r$ $\quad$ m, t free

b. (*2 pts*) Explain what must be true of m, n, r and t for BA to exist.

$t = m$ $\quad$ n, r free

c. (*4 pts*) Explain what must be true of m and n for A^2 to exist. What about for A^T to exist?

$m = n$ for A^2 to exist.

m and n are free for A^T to exist.

d. (*2 pts*) Suppose $T(\vec{x}) = A\vec{x}$. T maps $\mathbb{R}^? \to \mathbb{R}^\star$. What is ? and what is $\star$?.

$? = n$ $\quad$ $\star = m$

8. (*10 points*) Find all a,b,c $\in \mathbb{R}$ (a,b,c $\neq 0$) for which the matrix $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ solves the equation $A^2 - a * A^T = I_2$ where I_2 is the 2×2 identity matrix.

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} - a\begin{bmatrix} a & c \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2+bc & ab \\ ac & cb \end{bmatrix} - \begin{bmatrix} a^2 & ac \\ ab & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} bc & ab-ac \\ ac-ab & cb \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$bc = 1$

$\left.\begin{array}{l} ab - ac = 0 \Rightarrow a(b-c) = 0 \\ ac - ab = 0 \Rightarrow a(c-b) = 0 \end{array}\right\}$ Since $a \neq 0$ $\quad (b-c) = 0 \Rightarrow b = c$

also $bc = 1$

$\therefore$ $b = 1$ and $c = 1$

or

$b = -1$ and $c = -1$

a is any non-zero real #

Solutions:

$$A = \left\{ \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & -1 \\ -1 & 0 \end{pmatrix} \;\middle|\; a \in \mathbb{R}_{\neq 0} \right\}$$