Answer Key for Exam #1

1. The vectors $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ have lengths $\sqrt{1^2+1^2+1^2+1^2} = 2$ and $\sqrt{1^2+1^2+0^2+1^2} = \sqrt{3}$, respectively, and their dot product is $1+1+0+1=3$. So the angle θ between them satisfies $\cos\theta = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$. Therefore $\theta = \frac{\pi}{6}$, or $30°$.

2. We use elimination on an augmented matrix. Subtracting the first row from the second and third, and then adding the second row to the third, we have

$$\begin{pmatrix}1&1&2&&3\\1&1&3&&7\\1&9&1&&15\end{pmatrix} \longrightarrow \begin{pmatrix}1&1&2&&3\\0&0&1&&4\\0&8&-1&&12\end{pmatrix} \longrightarrow \begin{pmatrix}1&1&2&&3\\0&0&1&&4\\0&8&0&&16\end{pmatrix}.$$

The last step wasn't necessary since the system was already triangular, but it doesn't hurt. The second equation gives $x_3 = 4$ and the third one says $x_2 = 2$. Then the first equation becomes $x_1 + 2 + 8 = 3$, so $x_1 = -7$.

3. Here we have to reduce $[A \;\; I]$ to $\left[I \;\; A^{-1}\right]$ by row operations. Using the first pivot to eliminate the other entries in the first column of A we have

$$\begin{pmatrix}1&2&3&&1&0&0\\2&3&1&&0&1&0\\3&1&2&&0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&3&&1&0&0\\0&-1&-5&&-2&1&0\\0&-5&-7&&-3&0&1\end{pmatrix}.$$

Multiplying the second row by -1 to make the second pivot equal 1, and then using it to clear out the other entries in the second column, we get

$$\begin{pmatrix}1&2&3&&1&0&0\\0&-1&-5&&-2&1&0\\0&-5&-7&&-3&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&-7&&-3&2&0\\0&1&5&&2&-1&0\\0&0&18&&7&-5&1\end{pmatrix}.$$

To put off the fractions as long as possible we can multiply the first and second rows by 18:

$$\begin{pmatrix}1&0&-7&&-3&2&0\\0&1&5&&2&-1&0\\0&0&18&&7&-5&1\end{pmatrix} \longrightarrow \begin{pmatrix}18&0&-126&&-54&36&0\\0&18&90&&36&-18&0\\0&0&18&&7&-5&1\end{pmatrix}.$$

If we now add 7 times the third row to the first, and subtract 5 times the third row from the second, we get

$$\begin{pmatrix}18&0&-126&&-54&36&0\\0&18&90&&36&-18&0\\0&0&18&&7&-5&1\end{pmatrix} \longrightarrow \begin{pmatrix}18&0&0&&-5&1&7\\0&18&0&&1&7&-5\\0&0&18&&7&-5&1\end{pmatrix}.$$

Therefore $A^{-1} = \dfrac{1}{18}\begin{pmatrix}-5&1&7\\1&7&-5\\7&-5&1\end{pmatrix}$.

4(a) Step 1: subtract twice the first row of A from the second, and thrice the first row from the third. Step 2: subtract 4 times the second row from the third:

$$A = \begin{pmatrix}1&4&3\\2&9&8\\3&16&18\end{pmatrix} \longrightarrow \begin{pmatrix}1&4&3\\0&1&2\\0&4&9\end{pmatrix} \longrightarrow \begin{pmatrix}1&4&3\\0&1&2\\0&0&1\end{pmatrix} = U.$$

Step 1 puts a 2 and 3 respectively in the second and third rows of the first column of L, and step 2 puts a 4 in the middle of the bottom row of L, so $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$ and $A = LU$.

4(b) We may now solve $A\vec{x} = \begin{pmatrix} 7 \\ 13 \\ 15 \end{pmatrix}$ in two steps. First we solve $L\vec{c} = \vec{b}$, which in this case is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 15 \end{pmatrix},$$

and we get $c_1 = 7$; $14 + c_2 = 13$, so $c_2 = -1$; and $21 - 4 + c_3 = 15$ so $c_3 = -2$. Finally we solve $U\vec{x} = \vec{c}$, which in this case is

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -2 \end{pmatrix}.$$

Here we have $x_3 = -2$; $x_2 - 4 = -1$ so $x_2 = 3$; $x_1 + 12 - 6 = 7$ so $x_1 = 1$.

4(c) Here we have to reduce $[L \;\; I]$ to $\left[I \;\; L^{-1}\right]$ by row operations, which takes three easy steps:

$$\begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 2 & 1 & 0 & & 0 & 1 & 0 \\ 3 & 4 & 1 & & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -2 & 1 & 0 \\ 0 & 4 & 1 & & -3 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 1 & & 5 & -4 & 1 \end{pmatrix}.$$

So $\quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$, and by symmetry $\quad U^{-1} = \begin{pmatrix} 1 & -4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

4(d) Since $A = LU$ and L and U are both invertible, we have $A^{-1} = U^{-1}L^{-1}$. Therefore

$$\text{if} \quad A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 9 & 8 \\ 3 & 16 & 18 \end{pmatrix} \quad \text{then} \quad A^{-1} = \begin{pmatrix} 1 & -4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 34 & -24 & 5 \\ -12 & 9 & -2 \\ 5 & -4 & 1 \end{pmatrix}.$$

5(i) A large family of examples is

$$\begin{pmatrix} ra & rb \\ sa & sb \end{pmatrix} \begin{pmatrix} ub & vb \\ -ua & -va \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} ub & vb \\ -ua & -va \end{pmatrix} \begin{pmatrix} ra & rb \\ sa & sb \end{pmatrix} = (ru + sv) \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

which is not O unless $ru + sv = 0$ or a and b are both 0. A specific example is

$$\begin{pmatrix} 12 & 20 \\ 9 & 15 \end{pmatrix} \begin{pmatrix} 20 & -25 \\ -12 & 15 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 20 & -25 \\ -12 & 15 \end{pmatrix} \begin{pmatrix} 12 & 20 \\ 9 & 15 \end{pmatrix} = \begin{pmatrix} 15 & 25 \\ -9 & -15 \end{pmatrix}.$$

5(ii) If A is invertible then apply A^{-1} to $AB = O$ on the left to get $A^{-1}AB = A^{-1}O$, which simplifies to $B = O$. But if $B = O$, then $BA = OA = O$. Similarly, if B is invertible then apply B^{-1} to $AB = O$ on the right to get $ABB^{-1} = OB^{-1}$, which simplifies to $A = O$. If $A = O$, then $BA = BO = O$. So if either A or B is invertible, then $AB = O$ implies $BA = O$ (and conversely).

5(iii) If we transpose the equation $AB = O$ we get (remember that the order switches) $B^T A^T = O^T$. But $O^T = O$, so this says $B^T A^T = O$. If A and B are symmetric then this becomes $BA = O$.

6. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $A^{-1} = A^T$, then we have $ad - bc \neq 0$ (otherwise A^{-1} does not exist) and

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

This gives the two pairs of equations

$$a = \frac{d}{ad - bc} \qquad c = \frac{-b}{ad - bc}$$
$$d = \frac{a}{ad - bc} \qquad b = \frac{-c}{ad - bc}$$

which we can rewrite as

$$d = a\ (ad - bc) \qquad -b = c\ (ad - bc)$$
$$a = d\ (ad - bc) \qquad -c = b\ (ad - bc)$$

If we substitute each equation into the other in each pair we get

$$d = d\ (ad - bc)^2 \qquad b = b\ (ad - bc)^2$$
$$a = a\ (ad - bc)^2 \qquad c = c\ (ad - bc)^2$$

Unless $(ad - bc)^2 = 1$, these equations can only hold if all four of a, b, c, d are 0, but in that case $ad - bc = 0$ and A^{-1} does not exist. So we must have either $ad - bc = 1$ or $ad - bc = -1$.

If $ad - bc = 1$ then the equations

$$d = a\ (ad - bc) \qquad -b = c\ (ad - bc)$$
$$a = d\ (ad - bc) \qquad -c = b\ (ad - bc)$$

become $d = a$ and $-b = c$. If we substitute these into $ad - bc = 1$ it becomes $a^2 + c^2 = 1$. So (a, c) is a point on the unit circle, and therefore there is an angle θ such that $a = \cos\theta$ and $c = \sin\theta$. Then $d = \cos\theta$ and $b = -\sin\theta$ and we have a rotation matrix.

The other possibility is $ad - bc = -1$. In this case the equations

$$d = a\ (ad - bc) \qquad -b = c\ (ad - bc)$$
$$a = d\ (ad - bc) \qquad -c = b\ (ad - bc)$$

reduce to $b = c$ and $d = -a$. Substituting these into $ad - bc = -1$ we get $-a^2 - c^2 = -1$, or $a^2 + c^2 = 1$. Again (a, c) is a point on the unit circle, and so there is an angle, which we may as well call 2ϕ, such that $a = \cos 2\phi$ and $c = \sin 2\phi$. Then $b = \sin 2\phi$ and $d = -\cos 2\phi$ and we have a reflection matrix.