MATH 205A,B LINEAR ALGEBRA - PROF. P. WONG

EXAM I - OCTOBER 7, 2015

NAME: **Section:**(Circle one) A(8 : 00) B(9 : 30)

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Consider the following system of linear equations

$$\begin{aligned} 4x_1 - 2x_2 + 7x_3 &= -5 \\ 8x_1 - 3x_2 + 10x_3 &= -3. \end{aligned} \tag{1}$$

(a) Find the solutions to the system (1), if it is consistent.

Consider the associted augmented matrix

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 7 & -5 \\ 0 & 1 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 7/4 & -5/4 \\ 0 & 1 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 & 9/4 \\ 0 & 1 & -4 & 7 \end{bmatrix}.$$

Thus we conclude that $x_1 - \frac{1}{4}x_3 = \frac{9}{4}$ **and** $x_2 - 4x_3 = 7$. **In other words,**

$$\begin{aligned} x_1 &= \frac{9}{4} + \frac{1}{4}x_3 \\ x_2 &= 7 + 4x_3 \\ x_3 &= \textbf{ free vairable} \end{aligned}$$

(b) Find the solutions to the homogeneous system

$$\begin{aligned} 4x_1 - 2x_2 + 7x_3 &= 0 \\ 8x_1 - 3x_2 + 10x_3 &= 0. \end{aligned}$$

Using the calculations in part (a), the augmented matrix of the homogeneous system is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}.$$

In other words,

$$\begin{aligned} x_1 &= \frac{1}{4}x_3 \\ x_2 &= 4x_3 \\ x_3 &= \textbf{ free vairable} \end{aligned}$$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T(x_1, x_2) = (3x_1 + 2x_2, -x_1 + 3x_2, x_1 + x_2).$$

(a) Find all $\vec{x}$ such that $T(\vec{x}) = \vec{0}$.

If $T(x_1, x_2) = \vec{0}$ then $-x_1 + 3x_2 = 0$ or $x_1 = 3x_2$. Moreover, $x_1 + x_2 = 0$ implies that $x_1 = -x_2$. It follows that $3x_2 = -x_2$ so $x_2 = 0$ and hence $x_1 = 0$. Hence, $T(x_1, x_2) = \vec{0}$ implies that $x_1 = 0 = x_2$.

(b) Determine whether T is one-to-one. Justify your answer.

From (a), $T(\vec{x}) = \vec{0}$ has only the trivial solution so T is one-to-one.

(c) Determine whether T is onto. Justify your answer.

First, the matrix of T is given by

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}$$

so that $T(\vec{x}) = A\vec{x}$. Then T is onto if and only if the columns of A span the codomain which is $\mathbb{R}^3$. The matrix A has only two columns so they cannot span $\mathbb{R}^3$. Thus, we conclude that T is NOT onto.

3. Let

$$B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 3 \end{bmatrix}.$$

(a) Are the columns of B linearly independent? Justify your answer.

Note that the third column is the sum of the first and second column. It follows that the columns are NOT linearly independent.

(b) Do the columns of B span $\mathbb{R}^2$? Justify your answer.

Observe that

$$B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus, $B\vec{x} = \vec{b}$ **always has a solution for any** $\vec{b}$ **or every row has a pivot. It follows that the columns of** B **span** $\mathbb{R}^2$**.**

(c) Write a formula for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ so that $T(\vec{x}) = B\vec{x}$ for any vector $\vec{x}$ in $\mathbb{R}^3$.

If $T(\vec{x}) = B\vec{x}$ **then**

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 - x_3, -2x_1 + 5x_2 + 3x_3).$$

4. Use elementary row operations to find the inverse A^{-1} of the following invertible matrix

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix}.$$

(Show all your steps.)

Note that

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 2 & 4 & 11 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 9 & -15 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

It follows that the inverse of A is given by

$$A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 5 \\ 0 & 1 & -2 \end{bmatrix}.$$

5. (a) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation given by

$$S(x, y) = (1 - xy, x + y).$$

Determine whether S is a **linear** transformation. Explain.

For S to be linear, $S(k\vec{u}) = kS(\vec{u})$ for any vector $\vec{u}$ and any constant k. Take $\vec{u} = (2, 2)$ and $k = 2$. Then

$$S(2(2, 2)) = S(4, 4) = (1 - 16, 4 + 4) = (-15, 8).$$

On the other hand,

$$2S((2, 2)) = 2(1 - 4, 2 + 2) = (-6, 8) \neq (-15, 8) = S(2(2, 2)).$$

Hence, S is NOT linear.

(b) Let $A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$. Find (i) AB; (ii) BA^T; and (iii) $A - B^T$.

First,

$$A^T = \begin{bmatrix} 1 & -4 \\ -3 & 1 \\ 0 & 1 \end{bmatrix} \qquad \textbf{and} \qquad B^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It follows that

$$AB = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}.$$

Similarly,

$$A^T B^T = \begin{bmatrix} 1 & -4 \\ -3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 1 & -5 \\ 1 & -3 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$A - B^T = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ -5 & 1 & 0 \end{bmatrix}.$$