TEST 1

Math 205
10/7/13

Name: $\underbrace{KEY}$
by writing my name I swear by the honor code

Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can be reduced into RREF with calculator without showing steps unless otherwise indicated). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements. Put a smiley face next to your name for one point.
- This test has 7 problems and one Bonus and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*18 points*) Consider the following matrix equation.

$$\begin{pmatrix} 3 & 5 & 4 & 0 & 1 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 8 \\ 0 \end{pmatrix}$$

a. (*5 pts*) Write the system as a vector equation.

$$x \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 8 \\ 0 \end{pmatrix}$$

b. (*5 pts*) Write the equation as a system of linear equations.

$$\begin{aligned} 3x + 5y + 4z + v &= -1 \\ 3z + 2u &= -3 \\ u + v &= 8 \\ -5v &= 0 \end{aligned}$$

c. (*8 pts*) Solve the system and use vector parametric form for your solution.

$$\left(\begin{array}{ccccc|c} 3 & 5 & 4 & 0 & 1 & 1 \\ 0 & 0 & 3 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 5/3 & 0 & 0 & 0 & 79/9 \\ 0 & 0 & 1 & 0 & 0 & -19/3 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left\{ t \begin{pmatrix} -5/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 79/9 \\ 0 \\ -19/3 \\ 8 \\ 0 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

2. (*10 points*) True or False. No explanation necessary.

- False If a system of linear equations has no free variables, then it has a unique solution.
- FalseEvery matrix is row equivalent to a unique matrix in row echelon form.
- TrueEvery matrix is row equivalent to a unique matrix in reduced row echelon form.
- False If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$.
- True If A is an $n \times (n+1)$ matrix with n pivots, then the columns of A span $\mathbb{R}^n$.

3. (*17 points*) Determine whether the following sets of vectors are linearly independent or linearly dependent. You should be able to do this by inspection. Give a brief explanation.

a. (*3 pts*) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \right\}.$

Dependent. One vector is a multiple of another.

b. (*3 pts*) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}.$

Dependent. Three vectors in a two dimensional space.

c. (*3 pts*) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$

Independent. Neither vector is a multiple of the other in two dimensional space.

d. (*2 pts*) Do the vectors in the set in part a. span $\mathbb{R}^3$?

No.

e. (*2 pts*) Do the vectors in the set in part b. span $\mathbb{R}^2$?

Yes.

f. (*2 pts*) Do the vectors in the set in part c. span $\mathbb{R}^2$?

Yes.

g. (*2 pts*) If we add the zero vector to the set in part c., does the set span $\mathbb{R}^2$?

Yes.

4. (*18 points*) Let $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 17 \\ \frac{5}{2} \\ -9 \\ -2 \end{pmatrix} \right\}$. If necessary, we can call the vectors in S, $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$, respectively.

a. (*6 pts*) The vector $\vec{0}$ is always a linear combination of vectors in S because we can use the scalar 0 as the weight for every vector. Are there others (non-trivial) ways to write $\vec{0}$ as linear combination of vectors in S? If so, find all the solutions. If not, explain.

$$\left(\begin{array}{ccc|c} 1 & 5 & 17 & 0 \\ 2 & 1 & 5/2 & 0 \\ 4 & -2 & -9 & 0 \\ -3 & -1 & -2 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

All the solutions to $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$ have vector $\vec{c}$ in the set:

$$\left\{ t \begin{pmatrix} 1/2 \\ -7/2 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

b. (*5 pts*) Express a vector in S as a linear combination of the other vectors or explain why this is impossible.

$$\vec{v_3} = \frac{-1}{2}\vec{v_1} + \frac{7}{2}\vec{v_2}$$

c. (*4 pts*) Given some vector $\vec{b}$ in $\mathbb{R}^4$, is every $\vec{b}$ a linear combination of the vectors in S? Explain.

No. These vectors do not span $\mathbb{R}^4$.

d. (*3 pts*) Geometrically, what is the span of the vectors in S?

The vectors span a plane in $\mathbb{R}^4$.

5. (*8 points*) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $T(\vec{x}) = A\vec{x}$.

a. (*2 pts*) The matrix A has dimensions $\underline{m} \times \underline{n}$.

b. (*2 pts*) What may be true of m and n if T is onto? Circle all that apply.

$m > n$ $\quad\quad$ $\boxed{m < n}$ $\quad\quad$ $\boxed{\text{m=n}}$

c. (*2 pts*) What may be true of m and n if T is one-to-one? Circle all that apply.

$\boxed{m > n}$ $\quad\quad$ $m < n$ $\quad\quad$ $\boxed{\text{m=n}}$

d. (*2 pts*) What may be true of m and n if T is both one-to-one and onto? Circle all that apply.

$m > n$ $\quad\quad$ $m < n$ $\quad\quad$ $\boxed{\text{m=n}}$

6. (*20 points*) Let T and U be linear transformations.

$$T(x_1, x_2) = (x_2, x_1) \text{ and } U(x_1, x_2) = (x_1, 0)$$

a. (*8 pts*) Find the matrices A_T and A_U for the transformation T and U, respectively.

$$A_T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

b. (*4 pts*) Is T one-to-one? Is T onto? Briefly explain.

Yes, one-to-one, there is a pivot in every column. Yes, onto, there is a pivot in every column.

c. (*4 pts*) Is U one-to-one? Is U onto? Briefly explain.

No, not one-to-one, there is not a pivot in every column. No, not onto, there is not a pivot in every row.

d. (*4 pts*) Consider the transformation $(U + T)$. What is the image $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ under the transformation $(U + T)$?

$$(U+T)\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

7. (*8 points*) T is a linear transformation. $T:\mathbb{R}^{(2\times 2)}\to\mathbb{R}^{(2\times 2)}$. This means T maps 2×2 matrices to 2×2 matrices. Suppose

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}2&2\\1&1\end{bmatrix}, T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&1\end{bmatrix}, T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}1&1\\2&2\end{bmatrix}, T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

a. (*4 pts*) Write $\begin{bmatrix}3&2\\-1&3\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&0\end{bmatrix}$, and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$.

$$\begin{bmatrix}3&2\\-1&3\end{bmatrix}=3*\begin{bmatrix}2&2\\1&1\end{bmatrix}+2*\begin{bmatrix}0&0\\0&1\end{bmatrix}-1*\begin{bmatrix}1&1\\2&2\end{bmatrix}+3*\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

b. (*4 pts*) Determine $T\left(\begin{bmatrix}3&2\\-1&3\end{bmatrix}\right)$.

$$T\left(\begin{bmatrix}3&2\\-1&3\end{bmatrix}\right)=T\left(3*\begin{bmatrix}1&0\\0&0\end{bmatrix}+2*\begin{bmatrix}0&1\\0&0\end{bmatrix}-1*\begin{bmatrix}0&0\\1&0\end{bmatrix}+3*\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$$

$$=3*T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+2*T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)-1*T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)+3*T(\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$$

$$=3*\begin{bmatrix}2&2\\1&1\end{bmatrix}+2*\begin{bmatrix}0&0\\0&1\end{bmatrix}-1*\begin{bmatrix}1&1\\2&2\end{bmatrix}+3*\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$=\begin{bmatrix}6&6\\3&3\end{bmatrix}+\begin{bmatrix}0&0\\0&2\end{bmatrix}-\begin{bmatrix}1&1\\2&2\end{bmatrix}+\begin{bmatrix}3&0\\0&0\end{bmatrix}=\begin{bmatrix}8&5\\1&3\end{bmatrix}$$

Bonus Question (2 Extra Credit Points):

Suppose $a+b=\min(a,b)$ and $a*b=a+b$.
What is $(2+3)*((-1)+(-4))+(-4)^2$?

$$(2)*(-4)+-8=1+-8=-8$$

Scrap Page
(please do not remove this page from the test packet)