

This giant matrix....

exam02 cover info.txt

$$\begin{bmatrix} 1 & 4 & 23 & 20 & 20 & -4 & -4 \\ 3 & 13 & 73 & 65 & 65 & -19 & -19 \\ -5 & -17 & -102 & -88 & -88 & 15 & 15 \\ 1 & 1 & 15 & -7 & -8 & 81 & 80 \end{bmatrix}$$

is row equiv to this (but this is NOT QUITE in RREF)

$$\begin{bmatrix} 1 & 0 & 0 & 21 & 21 & -88 & -88 \\ 0 & 1 & 0 & 17 & 17 & -71 & -71 \\ 0 & 0 & 1 & -3 & -3 & 16 & 16 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Also, these are equivalent:

$$\begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix}$$

Useful info

MATH 205A EXAM II

Name:

suggested solutions

10/07/2005

- ① Show all your work
- ② be NEAT!
- ③ read the questions!

GOOD LUCK!

1
2
3
4
5
6I
6II
TOTAL

1. Suppose T and S are linear transformations and A and B are their corresponding matrices, where A and B are given below.

$$A = \begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 12 \end{bmatrix} \quad \begin{array}{c} T \\ \mathbb{R}^3 \rightarrow \mathbb{R}^3 \end{array} \quad \begin{array}{c} S \\ \mathbb{R}^3 \rightarrow \mathbb{R}^2 \end{array}$$

1a. $S: \mathbb{R}^k \rightarrow \mathbb{R}^m$ where k and m have what values? $k=3, m=2$

1b. Explain why you can't form the composition, $T \circ S$.

The "output" of S is vectors in \mathbb{R}^2 , yet the domain of T is \mathbb{R}^3

1c. Let c_{ij} denote the i - j th entry of the matrix of the composition $S \circ T$. Explicitly find c_{23} .

c_{23} is obtained by multiplying row 2 of B by column 3 of A :

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix} \xrightarrow{\text{row 2}} = 3(-7) + 6(-24) + 12 \cdot 27 = -21 - 144 + 324 = 159$$

1d. What if any conditions are there on $\mathbf{b}_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ so that \mathbf{b} is in the image (or range, as the book would say) of S , ie so that $S(\mathbf{v}) = \mathbf{b}$ for some \mathbf{v} in \mathbb{R}^3 ? Show all your work.

This is equivalent to asking,

"When does $B\vec{v} = \vec{b}$

have a soln \vec{v} ?" Row reduction yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 3 & 6 & 12 & b_2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 3 & b_2 - 3b_1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & \frac{b_2 - 3b_1}{3} \end{array} \right]$$

\therefore every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the range;

there are no inconsistencies when solving the matrix eqn $B\vec{v} = \vec{b}$

1e. Is S a 1 to 1 linear transformation? Explain fully.

NO. $S(\vec{v}) = \vec{0}$ has ∞ -many solns b/c

$B\vec{v} = \vec{0}$ has ∞ -many solns b/c

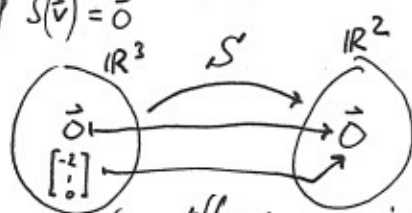
row reduction yields a free variable.

$$\left(\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 12 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \text{ shows } x_2 \text{ is free;} \right.$$

any vector of the form $x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ gives

a solution of $S(\vec{v}) = \vec{0}$

eg.:



(two different vectors in \mathbb{R}^3 are sent to $\vec{0}$ by S , so S is NOT 1-1.)

1f. Is S onto \mathbb{R}^m ? Explain fully.

YES: in 1d we saw that

Any $\vec{b} \in \mathbb{R}^2$ is in the image of S ,

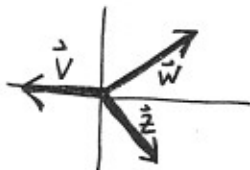
that is, S is onto \mathbb{R}^2

2a. Sketch two non-zero, unequal vectors in \mathbb{R}^2 which are not linearly independent, or explain why there are none.



\vec{v} and $2\vec{v}$ are non zero, unequal, and clearly $\{\vec{v}, 2\vec{v}\}$ is NOT a L.I. set.

2b. Sketch three different non-zero vectors in \mathbb{R}^2 which span \mathbb{R}^2 , or explain why there are none.



3 In all parts of this question, matrices are in $M_{3,3}$.

3a. What elementary matrix E_1 represents "adding 3 copies of row 2 to row 1"?

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3b. What elementary matrix E_2 represents "adding 1 copy of row 2 to row 3"?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

3c. What sequence of elementary row operations does the product $E_1 * E_2$ represent?

Given a matrix A , we'd write $E_1 E_2 A$. Then First, E_2 would add 1 copy of row 2 to row 3 of A , and with this new matrix, E_1 would then add 3 copies of row 2 to row 1.

3d. Does $E_2 * E_1$ represent the same sequence of elementary row operations? (Compute the two products. What do you learn?)

in general, $E_2 * E_1 \neq E_1 * E_2$ (matrix multiplication is NOT commutative)
and the SEQUENCE of operations is reversed ... but in this case,
since $E_2 * E_1$ does equal $E_1 * E_2$ (both are $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$)
the end result is the same, BUT THE SEQUENCE IS DIFFERENT.
(9 steps)

4a. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries and which is not invertible.

we need the determinant to be 0.

try $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ for example.

4b. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries such that the inverse also has four different, nonzero integer entries.

we need the determinant to be 1:

for example if $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

5 What is the definition of Linear Independence? "A set $S = \{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if ..."

... the only linear combination of the vectors in S which equals $\vec{0}$ is the one in which all weights $\alpha_1, \alpha_2, \dots, \alpha_p = 0$

6 Consider the system of equations:

$$\begin{aligned}x_1 + 4x_2 + 23x_3 + 20x_4 &= -4 \\3x_1 + 13x_2 + 73x_3 + 65x_4 &= -19 \\-5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\x_1 + x_2 + 15x_3 - 7x_4 &= 81\end{aligned}$$

Use the information on the front of the exam to answer these questions.

6a. Find the row reduced echelon form of the augmented matrix corresponding to this system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

6b. Write the complete set of solutions in the form $\mathbf{v}_h + \mathbf{p}$ as done in class, that is, where \mathbf{v}_h represents all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ and \mathbf{p} is a particular solution to $A\mathbf{x} = \mathbf{b}$.

we have $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -21 \\ -17 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -88 \\ -71 \\ 16 \\ 0 \end{bmatrix}$ where x_4 is free.

6c. If the 81 is changed just a little, to 80, what is the RREF of the resulting system? (you might have to do a little additional row reduction and you might not)

from the info, front cover:
an Echelon form is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$ which in RREF is $\left[\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$

6d. What is the complete set of solutions for this new system, again in the form $\mathbf{v}_h + \mathbf{p}$?

now the RREF reveals an inconsistency in the last row;
there can be NO SOLUTION

6e. Do the columns of the coefficient matrix A span \mathbb{R}^4 ? Explain.

NO. for example, Gol shows that $\begin{bmatrix} -4 \\ -19 \\ 15 \\ 80 \end{bmatrix}$ is not a L.C. of the columns of A , so those columns do not span all of \mathbb{R}^4

6f. Are they linearly independent? Explain.

NO. the free variable x_4 tells us that $A\vec{x} = \vec{0}$ has ∞ -many non trivial solutions, so it's possible to form a L.C. $x_1 \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 13 \\ -17 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 23 \\ 73 \\ -102 \\ 15 \end{bmatrix} + x_4 \begin{bmatrix} 20 \\ 65 \\ -88 \\ -7 \end{bmatrix}$ which equals $\vec{0}$ yet not all the weights must be 0

Problem 6 continues... for reference here's the original system:

$$\begin{aligned}x_1 + 4x_2 + 23x_3 + 20x_4 &= -4 \\3x_1 + 13x_2 + 73x_3 + 65x_4 &= -19 \\-5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\x_1 + x_2 + 15x_3 - 7x_4 &= 81\end{aligned}$$

6g. If we keep the 81, but change the -7 just a little to -8, what is the RREF of the resulting system? (you might have to do a little further row reduction and you might not)

from the front
cover, an
ECHELON form is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \text{ which is RREF } \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -88 \\ 0 & 1 & 0 & 0 & -71 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

6h. Do the columns of the corresponding new coefficient matrix B in part (6g) span \mathbb{R}^4 ? Explain.

YES! because now, for any $\vec{b} \in \mathbb{R}^4$, the matrix equation $B\vec{x} = \vec{b}$ always has at least one solution; there can NOT be any inconsistencies no matter what \vec{b} is.

6i. Is this set of columns linearly independent? Explain.

YES! because there are no free variables, the matrix equation $B\vec{x} = \vec{0}$ has ONLY the trivial soln, i.e., the only L.C. $x_1 \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 13 \\ -17 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 23 \\ 73 \\ -102 \\ 15 \end{bmatrix} + x_4 \begin{bmatrix} 20 \\ 65 \\ -88 \\ -8 \end{bmatrix}$ which

6j. Is either A or B invertible?

A is not since $\text{RREF}(A)$ is not I_4 .

B is since $\text{RREF}(B)$ is I_4 ,

yields the zero vector is the one in which all four weights x_1, x_2, x_3 and x_4 MUST be 0.