This giant matrix....

$$\begin{bmatrix} 1 & 4 & 23 & 20 & 20 & -4 & -4 \\ 3 & 13 & 73 & 65 & 65 & -19 & -19 \\ -5 & -17 & -102 & -88 & -88 & 15 & 15 \\ 1 & 1 & 15 & -7 & -8 & 81 & 80 \end{bmatrix}$$

is row equiv to this (but this is NOT QUITE in RREF)

$$\begin{bmatrix} 1 & 0 & 0 & 21 & 21 & -88 & -88 \\ 0 & 1 & 0 & 17 & 17 & -71 & -71 \\ 0 & 0 & 1 & -3 & -3 & 16 & 16 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Also, these are equivalent:

$$\begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix}$$

useful info

MATH 205_A EXAM II

1	
2	
3	
4	
5	
6 I	
6 II	
TOTAL	

Name:

suggested solutions

10/07/2005

1. Show all your work
2. be NEAT!
3. read the questions!

GOOD LUCK!

1. Suppose T and S are linear transformations and A and B are their corresponding matrices, where A and B are given below.

$$A = \begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 12 \end{bmatrix}$$

T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ S: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

1a. $S : \mathbf{R}^k \to \mathbf{R}^m$ where k and m have what values? $\boxed{K=3}$, $\boxed{m=2}$

1b. Explain why you can't form the composition, $T \circ S$.

The "output" of S is vectors in $\mathbb{R}^2$, yet the domain of T is $\mathbb{R}^3$

1c. Let c_{ij} denote the i-j^{th} entry of the matrix of the composition $S \circ T$. Explicitly find c_{23}.

C_{23} is obtained by multiplying row 2 of B by column 3 of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 12 \end{bmatrix}\begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix} \longrightarrow = 3(-7)+6(-24)+12\cdot 27 = -21-144+324 = \boxed{159}$$

no 2

1d. What if any conditions are there on $\mathbf{b}_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ so that $\mathbf{b}$ is in the image (or range, as the book would say) of S, ie so that $S(\mathbf{v}) = \mathbf{b}$ for some $\mathbf{v}$ in $\mathbf{R}^k$? Show all your work.

This is equivalent to asking, "when does $B\vec{v} = \vec{b}$ have a soln $\vec{v}$" Row reduction yields

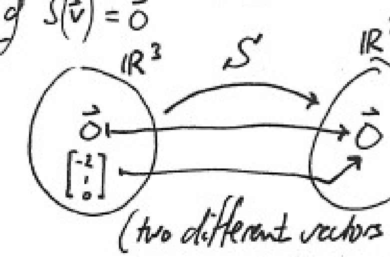
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 3 & 6 & 12 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 3 & b_2-3b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & \frac{b_2-3b_1}{3} \end{array}\right]$$

$\therefore$ every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the range; there are no inconsistencies when solving the matrix eqn $B\vec{v} = \vec{b}$

1e. Is S a 1 to 1 linear transformation? Explain fully.

__NO__. $S(\vec{v}) = \vec{0}$ has ∞-many soln's b/c $B\vec{v} = \vec{0}$ has ∞-many soln's b/c row reduction yields a free variable. ($\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 12 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ shows x_2 is free; any vector of the form $x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ gives a solution of $S(\vec{v}) = \vec{0}$

e.g. :



(two different vectors in $\mathbb{R}^3$ are sent to $\vec{0}$ by S, so S is NOT 1-1.)

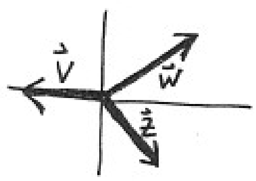
1f. Is S onto $\mathbf{R}^m$? Explain fully.

__YES__: in 1d we saw that Any $\vec{b} \in \mathbb{R}^2$ is in the image of S, that is, S is onto $\mathbb{R}^2$

2a. Sketch two non-zero, unequal vectors in $\mathbf{R}^2$ which are not linearly independent, or explain why there are none.

$\vec{v}$ and $2\vec{v}$ are non zero, unequal, and clearly $\{\vec{v}, 2\vec{v}\}$ is <u>not</u> a L.I. set.

2b. Sketch three different non-zero vectors in $\mathbf{R}^2$ which span $\mathbf{R}^2$, or explain why there are none.



3 In all parts of this question, matrices are in $M_{3,3}$.

3a. What elementary matrix E_1 represents "adding 3 copies of row 2 to row 1"?

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3b. What elementary matrix E_2 represents "adding 1 copies of row 2 to row 3"?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

3c. What sequence of elementary row operations does the product $E_1 * E_2$ represent?

Given a matrix A, we'd write $E_1 E_2 A$. Then

<u>First</u>, E_2 would add 1 copy of row 2 to row 3 of A, and with this new matrix, E_1 would then add 3 copies of row 2 to row 1.

3d. Does $E_2 * E_1$ represent the *same* sequence of elementary row operations? (Compute the two products. What do you learn?)

in <u>general</u>, $E_2 * E_1 \neq E_1 * E_2$ (matrix multiplication is <u>NOT</u> commutative) and the <u>SEQUENCE</u> of operations <u>is</u> <u>reversed</u> ... but in this case, since $E_2 * E_1$ <u>does</u> equal $E_1 * E_2$ (both are $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$) the <u>end result</u> is the same. BUT THE <u>SEQUENCE</u> IS DIFFERENT.

(of steps)

4a. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries and which is not invertible.

we need the determinant to be 0.

try $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ for example.

4b. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries such that the inverse also has four different, nonzero *integer* entries.

we need the determinant to be 1:

for example if $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ then $A^{-1} = \frac{1}{\det A}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

5 What is the definition of Linear Independence? "A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if ..."

... the <u>only</u> Linear combination of the vectors in S

$\alpha\vec{v}_1 + \alpha\vec{v}_2 + \cdots + \alpha_p\vec{v}$ which equals $\vec{0}$

is the one in which all weights $\alpha_1, \alpha_2, \ldots, \alpha_p = 0$

6 Consider the system of equations:

$$\begin{aligned} x_1 + 4x_2 \quad +23x_3 + 20x_4 &= -4 \\ 3x_1 + 13x_2 \quad +73x_3 + 65x_4 &= -19 \\ -5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\ x_1 + x_2 \quad +15x_3 - 7x_4 &= 81 \end{aligned}$$

Use the information on the front of the exam to answer these questions.

6a. Find the row reduced echelon form of the augmented matrix corresponding to this system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

6b. Write the complete set of solutions in the form $\mathbf{v}_h + \mathbf{p}$ as done in class, that is, where $\mathbf{v}_h$ represents all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ and $\mathbf{p}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$.

we have $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -21 \\ -17 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -88 \\ -71 \\ 16 \\ 0 \end{bmatrix}$ where x_4 is free.

6c. If the 81 is changed just a little, to 80, what is the RREF of the resulting system? (you might have to do a little additional row reduction and you might not)

from the info, front cover: an Echelon form is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ which in RREF is $\left[\begin{array}{cccc|c} & & & & 0 \\ & \text{same} & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

6d. What is the complete set of solutions for this new system, again in the form $\mathbf{v}_h + \mathbf{p}$?

now the RREF reveals an inconsistency in the last row;
there can be NO SOLUTION

6e. Do the columns of the coefficient matrix A span $\mathbf{R}^4$? Explain.

NO. for example, 6d shows that $\begin{bmatrix} -4 \\ -19 \\ 15 \\ 80 \end{bmatrix}$ is not a L.C. of the columns of A, so those columns do not span all of R^4

6f. Are they linearly independent? Explain.

NO. the free variable x_4 tells us that
$A\vec{x} = \vec{0}$ has ∞-many non trivial solutions, so
it's possible to form a L.C. $x_1 \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 13 \\ -17 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 23 \\ 73 \\ -102 \\ 15 \end{bmatrix} + x_4 \begin{bmatrix} 20 \\ 65 \\ -88 \\ -7 \end{bmatrix}$ which equals $\vec{0}$ yet not all the weights must be 0

Problem 6 continues... for reference here's the original system:

$$\begin{aligned} x_1 + 4x_2 \quad +23x_3 + 20x_4 &= -4 \\ 3x_1 + 13x_2 \ +73x_3 + 65x_4 &= -19 \\ -5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\ x_1 + x_2 \quad +15x_3 - 7x_4 &= 81 \end{aligned}$$

6g. If we keep the 81, but change the -7 just a little to -8, what is the RREF of the resulting system? (you might have to do a little further row reduction and you might not)

from the front cover, an ECHELON form is: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 21 & -88 \\ 0 & 1 & 0 & 17 & -71 \\ 0 & 0 & 1 & -3 & 16 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ which is RREF $\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -88 \\ 0 & 1 & 0 & 0 & -71 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

6h. Do the columns of the corresponding new coefficient matrix B in part (6g) span $\mathbf{R}^4$? Explain.

YES! because now, for any $\vec{b} \in \mathbb{R}^4$, the matrix equation $B\vec{x} = \vec{b}$ always has at least one solution; there can NOT be any inconsistencies no matter what $\vec{b}$ is.

6i. Is this set of columns linearly independent? Explain.

YES! because there are no free variables, the matrix equation $B\vec{x} = \vec{0}$ has ONLY the trivial sol'n, i.e., the only L.C. $x_1\begin{bmatrix}1\\3\\-5\\1\end{bmatrix} + x_2\begin{bmatrix}4\\13\\-17\\1\end{bmatrix} + x_3\begin{bmatrix}23\\73\\-102\\15\end{bmatrix} + x_4\begin{bmatrix}20\\65\\-88\\-8\end{bmatrix}$ which yields the zero vector is the one in which all four weights x_1, x_2, x_3 and x_4 MUST be 0.

6j. Is either A or B invertible?

A is not since RREF(A) is not I_4.

B is since RREF(B) is I_4.