1. Suppose T and S are linear transformations and A and B are their corresponding matrices, where A and B are given below.

$$A = \begin{bmatrix} 1 & 5 & -7 \\ 4 & 21 & -24 \\ -2 & -7 & 27 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 12 \end{bmatrix}$$

1a. $S : \mathbf{R}^k \to \mathbf{R}^m$ where k and m have what values?

1b. Explain why you can't form the composition, $T \circ S$.

1c. Let c_{ij} denote the i-j^{th} entry of the matrix of the composition $S \circ T$. Explicitly find c_{23}.

1d. What if any conditions are there on $\mathbf{b}_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ so that $\mathbf{b}$ is in the image (or range, as the book would say) of S, ie so that $S(\mathbf{v}) = \mathbf{b}$ for some $\mathbf{v}$ in $\mathbf{R}^k$? Show all your work.

1e. Is S a 1 to 1 linear transformation? Explain fully.

1f. Is S onto $\mathbf{R}^m$? Explain fully.

2a. Sketch two non-zero, unequal vectors in $\mathbf{R}^2$ which are not linearly independent, or explain why there are none.

2b. Sketch three different non-zero vectors in $\mathbf{R}^2$ which span $\mathbf{R}^2$, or explain why there are none.

3 In all parts of this question, matrices are in $M_{3,3}$.

3a. What elementary matrix E_1 represents "adding 3 copies of row 2 to row 1"?

3b. What elementary matrix E_2 represents "adding 1 copies of row 2 to row 3"?

3c. What sequence of elementary row operations does the product $E_1 * E_2$ represent?

3d. Does $E_2 * E_1$ represent the *same* sequence of elementary row operations? (Compute the two products. What do you learn?)

4a. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries and which is not invertible.

4b. Give an example of a matrix in $M_{2,2}$ which has four different, nonzero integer entries such that the inverse also has four different, nonzero *integer* entries.

5 What is the definition of Linear Independence? "A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if ..."

6 Consider the system of equations:

$$\begin{aligned} x_1 + 4x_2 \; +23x_3 + 20x_4 &= -4 \\ 3x_1 + 13x_2 \; +73x_3 + 65x_4 &= -19 \\ -5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\ x_1 + x_2 \; +15x_3 - 7x_4 &= 81 \end{aligned}$$

Use the information on the front of the exam to answer these questions.

6a. Find the row reduced echelon form of the augmented matrix corresponding to this system.

6b. Write the complete set of solutions in the form $\mathbf{v}_h + \mathbf{p}$ as done in class, that is, where $\mathbf{v}_h$ represents all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ and $\mathbf{p}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$.

6c. If the 81 is changed just a little, to 80, what is the RREF of the resulting system? (you might have to do a little additional row reduction and you might not)

6d. What is the complete set of solutions for this new system, again in the form $\mathbf{v}_h + \mathbf{p}$?

6e. Do the columns of the coefficient matrix A span $\mathbf{R}^4$? Explain.

6f. Are they linearly independent? Explain.

Problem 6 continues. . . for reference here's the original system:

$$\begin{aligned} x_1 + 4x_2 \quad +23x_3 + 20x_4 &= -4 \\ 3x_1 + 13x_2 \ +73x_3 + 65x_4 &= -19 \\ -5x_1 - 17x_2 - 102x_3 - 88x_4 &= 15 \\ x_1 + x_2 \quad +15x_3 - 7x_4 &= 81 \end{aligned}$$

6g. If we keep the 81, but change the -7 just a little to -8, what is the RREF of the resulting system? (you might have to do a little further row reduction and you might not)

6h. Do the columns of the corresponding new coefficient matrix B in part (6g) span $\mathbf{R}^4$? Explain.

6i. Is this set of columns linearly independent? Explain.

6j. Is either A or B invertible?