Name: Solutions

- Check that you have 7 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (10 points) (For this problem do all calculations by hand.) Determine all possible values of \( h \) and \( k \) such that the solution set of the following system

   (a) is empty (b) contains infinitely many solutions

   \[
   2x_1 + hx_2 = 5 \\
   4x_1 + 3x_2 = k
   \]

   Augmented matrix is
   \[
   \left[ \begin{array}{ccc|c}
   2 & h & 5 \\
   4 & 3 & k
   \end{array} \right] \sim \left[ \begin{array}{ccc|c}
   2 & h & 5 \\
   0 & 3-2h & k-10
   \end{array} \right]
   \]

   (a) Solution set is empty if \( 3-2h = 0 \) ie \( h = \frac{3}{2} \) and \( k-10 \neq 0 \) if \( k \neq 10 \).
   (In this case, reduced matrix looks like \( \left[ \begin{array}{ccc|c}
   2 & h & 5 \\
   0 & 0 & \text{non-zero \#}
   \end{array} \right] \).)

   (b) \( 3-2h = 0 \) if \( h = \frac{3}{2} \) and \( k-10 = 0 \) ie \( k = 10 \). (\( \left[ \begin{array}{ccc|c}
   2 & h & 5 \\
   0 & 0 & 0
   \end{array} \right] \).)

   In this case, we have one free variable and so infinitely many solutions.

2. (8 points) Let \( A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4] \) be a \( 3 \times 4 \) matrix. Suppose \( x_1 = 3, x_2 = 4, x_3 = 1, x_4 = 2 \) is a solution of the equation \( A\vec{x} = \vec{0} \).

   (a) Are the columns of \( A \) linearly independent? Explain.

   Columns of \( A \) are not linearly independent because the equation \( A\vec{x} = \vec{0} \) has a non-trivial solution, namely \( \left[ \begin{array}{c}
   3 \\
   4 \\
   1
   \end{array} \right] \).

   (b) Write the vector \( \vec{a}_4 \) as a linear combination of the vectors \( \vec{a}_1, \vec{a}_2, \) and \( \vec{a}_3 \).

   We have \( 3\vec{a}_1 + 4\vec{a}_2 + \vec{a}_3 + 2\vec{a}_4 = \vec{0} \)

   So \( 2\vec{a}_4 = -3\vec{a}_1 - 4\vec{a}_2 - \vec{a}_3 \)

   \( \vec{a}_4 = -3\vec{a}_1 - 4\vec{a}_2 - \frac{\vec{a}_3}{2} = -\frac{3}{2}\vec{a}_1 - 2\vec{a}_2 - \frac{1}{2}\vec{a}_3 \).
3. (10 points) Let \( A = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \\ 1 & 5 & 3 & 6 \end{bmatrix} \).

(a) Describe all solutions of \( A\bar{x} = \bar{0} \) in parametric vector form.

\[
\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \\ 1 & 5 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\sim \begin{bmatrix} 1 & 0 & -4.5 & -19 & 0 \\ 0 & 1 & 1.5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

So \( x_1 = 4.5x_3 + 19x_4 \), \( x_2 = -1.5x_3 - 5x_4 \), \( x_3, x_4 \) free.

In parametric vector form, we have

\[
\begin{bmatrix} 4.5x_3 + 19x_4 \\ -1.5x_3 - 5x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4.5 \\ -1.5 \end{bmatrix} + x_4 \begin{bmatrix} 19 \\ -5 \end{bmatrix}
\]

(b) Use your answer in part (a) to give two non-zero vectors that are solutions of the equation \( A\bar{x} = \bar{0} \).

Choose values for \( x_3 \) and \( x_4 \) to pick two non-zero vectors. \( x_3 = 1, x_4 = 0 \) gives \( \begin{bmatrix} 4.5 \\ -1.5 \end{bmatrix} \), \( x_3 = 0, x_4 = 1 \) gives \( \begin{bmatrix} 19 \\ -5 \end{bmatrix} \).

4. (6 points) Let \( \bar{a}_1, \bar{a}_2, \bar{b} \) and \( \bar{c} \) be the vectors in \( \mathbb{R}^2 \) shown in the figure, and let \( A = [\bar{a}_1 \ \bar{a}_2] \).

(a) Is the vector \( \bar{c} \) in Span\{\( \bar{a}_1, \bar{a}_2 \)\}? Explain.

The vector \( \bar{c} \) is in Span \( \{\bar{a}_1, \bar{a}_2\} \) because \( \bar{c} \) is on the line spanned by the two vectors. In fact, \( \bar{c} \) is a multiple of \( \bar{a}_1 \) (also, a multiple of \( \bar{a}_2 \)).

(b) Does the equation \( A\bar{x} = \bar{b} \) have a solution? Explain.

Since \( \bar{b} \) is not on the line spanned by the vectors \( \bar{a}_1, \bar{a}_2 \), so \( \bar{b} \) cannot be written as a linear combination of the vectors \( \bar{a}_1 \) and \( \bar{a}_2 \). Hence, the equation \( A\bar{x} = \bar{b} \) does not have a solution.
5. (8 points) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2, x_3) = (-2x_2 + 3x_3, 5x_1 - x_3)$.

(a) Find the standard matrix of $T$.

$$
\text{Standard matrix of } T : \begin{bmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{bmatrix}
= \begin{bmatrix}
0 & -2 & 3 \\
5 & 0 & -1
\end{bmatrix}.
$$

(b) Is $T$ one-to-one? Explain.

The matrix of $T$ is a $2 \times 3$ matrix. So it has a maximum of 2 pivots. So at least one column does not have a pivot. So there is at least one free variable. Hence, the equation $T(x) = \mathbf{0}$ has infinitely many solutions and so $T$ is not one-to-one.

6. (6 points) Suppose a $4 \times 4$ matrix $A$ is invertible. Explain (using pivots) why the columns of $A$ span $\mathbb{R}^4$. (If you find it helpful, you may use the following steps in your explanation.)

Since $A$ is invertible, $A$ reduces to $I_4$. So the number of pivots in $A$ is $4$.

Use this to explain why the columns of $A$ span $\mathbb{R}^4$.

$A$ has a pivot in each row.

Hence the columns of $A$ span $\mathbb{R}^4$. 

7. (12 points) Short answers: (No explanations needed. Simply write your answers. If you do some computation to get the answer, show the computation.)

(a) Suppose the vectors $\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3$ are in $\mathbb{R}^7$. How many vectors are in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

\[
\text{Infinitely many}
\]

(b) Suppose $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is an \textbf{onto} linear transformation. How many solutions does the equation $T(\vec{x}) = \begin{bmatrix} 80 \\ -45 \end{bmatrix}$ have?

\[
\text{Infinitely many}
\]

(c) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute $\vec{u}^T\vec{u}$.

- Compute $\vec{u}^T\vec{u}$.

\[
\vec{u}^T\vec{u} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

- Compute $\vec{u}^T\vec{u}$.

\[
\vec{u}^T\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5
\]

(d) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$, if it exists.

\[
\text{A does not reduce to the identity matrix.}
\]

\[
\text{So inverse of A does not exist.}
\]

(e) Let $T$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$, where $A$ is a $3 \times 5$ matrix. Suppose $T(\vec{u}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. Find $T(4\vec{u} - \vec{v})$.

\[
T(4\vec{u} - \vec{v}) = 4T(\vec{u}) - T(\vec{v}) = \begin{bmatrix} -4 \\ 13 \\ -7 \end{bmatrix}
\]