Math 205B Test 1 (60 points)

Name: Solutions

- Check that you have 7 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (10 points) (For this problem do all calculations by hand.) Determine all possible values of h and k such that the solution set of the following system

   (a) is empty (b) contains infinitely many solutions

$$\begin{aligned} 2x_1 + hx_2 &= 5 \\ 4x_1 + 3x_2 &= k \end{aligned}$$

Augmented matrix is $\begin{bmatrix} 2 & h & 5 \\ 4 & 3 & k \end{bmatrix} \overset{R2 = R2 - 2R1}{\sim} \begin{bmatrix} 2 & h & 5 \\ 0 & 3-2h & k-10 \end{bmatrix}$

(a) Solution set is empty if $3-2h=0$ ie $h=\frac{3}{2}$ and $k-10 \neq 0$ ie $k \neq 10$.

(In this case, reduced matrix looks like $\begin{bmatrix} 2 & h & 5 \\ 0 & 0 & \text{non-zero \#} \end{bmatrix}$.

(b) $3-2h=0$ ie $h=\frac{3}{2}$ and $k-10=0$ ie $k=10$. $\left(\begin{bmatrix} 2 & h & 5 \\ 0 & 0 & 0 \end{bmatrix}\right)$.

In this case, we have one free variable and so infinitely many solns.

2. (8 points) Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$ be a 3×4 matrix. Suppose $x_1 = 3, x_2 = 4, x_3 = 1, x_4 = 2$ is a solution of the equation $A\vec{x} = \vec{0}$.

   (a) Are the columns of A linearly independent? Explain.

   Columns of A are not linearly independent because the equation $A\bar{x}=\bar{0}$ has a non-trivial solution, namely $\begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$.

   (b) Write the vector $\vec{a}_4$ as a linear combination of the vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$.

   We have $3\bar{a}_1 + 4\bar{a}_2 + \bar{a}_3 + 2\bar{a}_4 = \bar{0}$

   So $2\bar{a}_4 = -3\bar{a}_1 - 4\bar{a}_2 - \bar{a}_3$

   $\bar{a}_4 = \dfrac{-3\bar{a}_1 - 4\bar{a}_2 - \bar{a}_3}{2} = -\frac{3}{2}\bar{a}_1 - 2\bar{a}_2 - \frac{1}{2}\bar{a}_3$

3. (10 points) Let $A = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \\ 1 & 5 & 3 & 6 \end{bmatrix}$.

(a) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form.

$$[A|\bar{0}] = \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \\ 1 & 5 & 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4.5 & -19 & 0 \\ 0 & 1 & 1.5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 4.5x_3 + 19x_4$
$x_2 = -1.5x_3 - 5x_4$
x_3, x_4 free

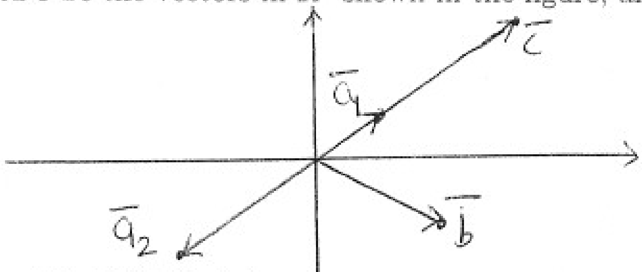
In parametric vector form, we have

$$\begin{bmatrix} 4.5x_3 + 19x_4 \\ -1.5x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4.5 \\ -1.5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 19 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

(b) Use your answer in part (a) to give two non-zero vectors that are solutions of the equation $A\vec{x} = \vec{0}$.

Choose values for x_3 and x_4 to pick two non-zero vectors.
$x_3 = 1$, $x_4 = 0$ gives $\begin{bmatrix} 4.5 \\ -1.5 \\ 1 \\ 0 \end{bmatrix}$. $x_3 = 0$, $x_4 = 1$ gives $\begin{bmatrix} 19 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

4. (6 points) Let $\vec{a}_1$, $\vec{a}_2$, $\vec{b}$ and $\vec{c}$ be the vectors in $\mathbb{R}^2$ shown in the figure, and let $A = [\vec{a}_1 \ \vec{a}_2]$.



(a) Is the vector $\vec{c}$ in Span$\{\vec{a}_1, \vec{a}_2\}$? Explain.

The vector $\bar{c}$ is in Span $\{\bar{a}_1, \bar{a}_2\}$ because $\bar{c}$ is on the line spanned by the two vectors. In fact, $\bar{c}$ is a multiple of $\bar{a}_1$. (also, a multiple of $\bar{a}_2$.)

(b) Does the equation $A\vec{x} = \vec{b}$ have a solution? Explain.

Since $\bar{b}$ is not on the line spanned by the vectors $\bar{a}_1, \bar{a}_2$, so $\bar{b}$ cannot be written as a linear combination of the vectors $\bar{a}_1$ and $\bar{a}_2$. Hence, the equation $A\bar{x} = \bar{b}$ does not have a solution.

5. (8 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2, x_3) = (-2x_2 + 3x_3, 5x_1 - x_3)$.

(a) Find the standard matrix of T.

Standard matrix of T: $[T(\bar{e}_1) \quad T(\bar{e}_2) \quad T(\bar{e}_3)]$

$$= \begin{bmatrix} 0 & -2 & 3 \\ 5 & 0 & -1 \end{bmatrix}.$$

(b) Is T one-to-one? Explain.

The matrix of T is a 2x3 matrix. So it has a maximum of 2 pivots. So atleast one column does not have a pivot. So there is atleast one free variable. Hence, the equation $T(\bar{x}) = \bar{0}$ has infinitely many solutions and so T is not one-to-one.

6. (6 points) Suppose a 4×4 matrix A is invertible. Explain (using pivots) why the columns of A span $\mathbb{R}^4$. (If you find it helpful, you may use the following steps in your explanation.)

Since A is invertible, A reduces to I_4. So the number of pivots in A is 4

Use this to explain why the columns of A span $\mathbb{R}^4$.

A has a pivot in each row.

Hence the columns of A span $\mathbb{R}^4$.

7. (12 points) Short answers: (No explanations needed. Simply write your answers. If you do some computation to get the answer, show the computation.)

(a) Suppose the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are in $\mathbb{R}^7$. How many vectors are in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

Infinitely many

(b) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^2$ is an **onto** linear transformation. How many solutions does the equation $T(\vec{x}) = \begin{bmatrix} 80 \\ -45 \end{bmatrix}$ have?

Infinitely many

(c) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\bar{u}^T = [1 \quad 2]$

- Compute $\vec{u}\vec{u}^T$.

$$\bar{u}\bar{u}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- Compute $\vec{u}^T\vec{u}$.

$$\bar{u}^T\bar{u} = [1 \quad 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [5]$$

(d) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$, if it exists.

A does not reduce to the identity matrix.
So inverse of A does not exist.

(e) Let T be a linear transformation given by $T(\vec{x}) = A\vec{x}$, where A is a 3×5 matrix. Suppose $T(\vec{u}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. Find $T(4\vec{u} - \vec{v})$.

$$T(4\bar{u} - \bar{v}) = 4T(\bar{u}) - T(\bar{v}) = \begin{bmatrix} -4 \\ 13 \\ -7 \end{bmatrix}$$