Answer Key for Exam #1

1. The vectors $\begin{pmatrix} 1 \\ 3 \\ -5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \\ 2 \end{pmatrix}$ have lengths 6 and 4, respectively:

$$\sqrt{1^2+3^2+(-5)^2+1^2} = \sqrt{36} = 6 \quad \text{and} \quad \sqrt{2^2+(-2)^2+2^2+2^2} = \sqrt{16} = 4.$$

Their dot product is $1\cdot 2 + 3(-2) + (-5)2 + 1\cdot 2 = -12$, so the angle θ between them satisfies $\cos\theta = \dfrac{-12}{6\cdot 4} = -\dfrac{1}{2}$. Therefore $\theta = 120°$, or $\theta = \frac{2\pi}{3}$.

2. We use elimination on an augmented matrix. First two steps: subtract three times the first row from the second, and four times the first row from the third. Next two steps: multiply the second row by -1, and then add twice the new second row to the third. This gives

$$\begin{pmatrix} 1 & 5 & 6 & 22 \\ 3 & 4 & 1 & 3 \\ 4 & -2 & 5 & 37 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 6 & 22 \\ 0 & -11 & -17 & -63 \\ 0 & -22 & -19 & -51 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 6 & 22 \\ 0 & 11 & 17 & 63 \\ 0 & 0 & 15 & 75 \end{pmatrix}.$$

The third equation gives $x_3 = 5$. Then the second equation becomes $11x_2 + 85 = 63$, so $x_2 = -2$. Finally the first equation becomes $x_1 - 10 + 30 = 22$, so $x_1 = 2$.

3(a) Step 1: subtract three times the first row of A from the second row, and subtract the first row from the third. Step 2: subtract twice the second row from the third.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 6 \\ 1 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = U.$$

Step 1 puts 3 in the second row, first column of L; and 1 in the third row, first column. Then step 2 puts a 2 in the second row, third column of L, so $L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

3(b) We may now solve $A\vec{x} = \begin{pmatrix} 5 \\ 8 \\ -12 \end{pmatrix}$ in two steps. First we solve $L\vec{y} = \vec{b}$, which in this case is

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ -12 \end{pmatrix}$$

and we get $y_1 = 5$; $15 + y_2 = 8$, so $y_2 = -7$; and $5 - 14 + y_3 = -12$, so $y_3 = -3$. Finally we solve $U\vec{x} = \vec{y}$, which in this case is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -3 \end{pmatrix}.$$

Here we have $x_3 = -3$; $x_2 - 9 = -7$, so $x_2 = 2$; $x_1 + 4 - 3 = 5$, so $x_1 = 4$.

3(c) Here we have to reduce $[L \;\; I]$ to $\left[I \;\; L^{-1}\right]$ by row operations, which is not too hard:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{pmatrix}.$$

So $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{pmatrix}$. From this it is a reasonable guess that $U^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$, since U is a sort of "double transpose" of L and this is what we get by transposing L^{-1} across both diagonals. One can easily check that

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

so this must indeed be U^{-1}. Alternatively we can reduce $[U \;\; I]$ to $\left[I \;\; U^{-1}\right]$ by row operations:

$$\begin{pmatrix} 1 & 2 & 1 & & 1 & 0 & 0 \\ 0 & 1 & 3 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -5 & & 1 & -2 & 0 \\ 0 & 1 & 3 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & & 1 & -2 & 5 \\ 0 & 1 & 0 & & 0 & 1 & -3 \\ 0 & 0 & 1 & & 0 & 0 & 1 \end{pmatrix}.$$

3(d) Since $A = LU$ and L and U are both invertible, we have $A^{-1} = U^{-1}L^{-1}$. Therefore

$$A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 32 & -12 & 5 \\ -18 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix}.$$

4(a) Step 1: subtract twice the first row of A from the second row, three times the first row from the third row, and 4 times the first row from the fourth row. Step 2: subtract twice the second row from the fourth.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 6 & 4 \\ 3 & 6 & 3 & 12 \\ 4 & 4 & 12 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -6 & 0 \\ 0 & -4 & 0 & -12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} = U.$$

Step 1 puts $2, 3, 4$ respectively in the second through fourth rows of column 1 of L, and step 2 puts a 2 in the second row, fourth column of L, so $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{pmatrix}$ and $A = LU$.

4(b) Since A is symmetric, we should be able to rewrite $U = DL^T$ where D is a diagonal matrix containing the pivots of A. Factoring $1, -2, -6, -4$ respectively from the rows of U we have

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \quad \text{and the remaining matrix is} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L^T.$$

4(c) To find D^{-1} we simply invert each number on the diagonal of D:

$$D^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

To find L^{-1} we have to reduce $[L \;\; I]$ to $\left[I \;\; L^{-1}\right]$ by row operations, which is not too hard:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 1 & & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & & -3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & & -4 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & & 0 & -2 & 0 & 1 \end{pmatrix}$$

So $L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$.

4(d) We have $M = LDL^T$, so $M^{-1} = \left(L^T\right)^{-1} D^{-1} L^{-1}$. We calculated D^{-1} and L^{-1} in 4(c), and $\left(L^T\right)^{-1} = \left(L^{-1}\right)^T$, so we have everything we need, and we just have to multiply these three matrices together:

$$\begin{aligned} M^{-1} &= \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{12} \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{12} \begin{pmatrix} 12 & 0 & 0 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & 0 & -2 & 0 \\ 0 & 6 & 0 & -3 \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} -30 & 12 & 6 & 0 \\ 12 & -18 & 0 & 6 \\ 6 & 0 & -2 & 0 \\ 0 & 6 & 0 & -3 \end{pmatrix}. \end{aligned}$$

5. We have

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

so we want to have u and v satisfying

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = u \begin{pmatrix} a & b \\ c & d \end{pmatrix} + v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the four equations

$$a^2 + bc = au + v, \qquad b(a+d) = bu, \qquad c(a+d) = cu, \qquad d^2 + bc = du + v.$$

To satisfy the second and third equations we must have $u = a + d$. Then we can find v from either the first or the fourth equation. The first one says that

$$v = a^2 + bc - au = a^2 + bc - a(a+d) = a^2 + bc - a^2 - ad = bc - ad,$$

and the fourth one says the same thing. Recalling that $ad - bc$ is the determinant of A (or $\det A$ for short) and $a + d$ the trace of A (or $\operatorname{tr} A$ for short), we have proved that if A is a 2×2 matrix, then $A^2 = (\operatorname{tr} A)\, A - (\det A)\, I$. A more complicated version of this is true for $n \times n$ matrices; it is called the Cayley-Hamilton theorem.

6. If we multiply $A^2 = (\operatorname{tr} A)\, A - (\det A)\, I$ by A^{-1} (assuming it exists) we get

$$A = (\operatorname{tr} A)\, I - (\det A)\, A^{-1}, \quad \text{or} \quad (\det A)\, A^{-1} = (\operatorname{tr} A)\, I - A.$$

Since by assumption A^{-1} exists, $\det A \neq 0$ so we can divide by it:

$$A^{-1} = \frac{\operatorname{tr} A}{\det A}\, I - \frac{1}{\det A}\, A.$$

Substituting in we get

$$A^{-1} = \frac{1}{\det A} \left\{ \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So problems 5 and 6 taken together have derived the 2×2 inverse formula.

7. A reflection matrix R satisfies $R^2 = I$, so the Cayley-Hamilton theorem holds with $u = 0$ and $v = 1$. A projection matrix P satisfies $P^2 = P$, so the Cayley-Hamilton theorem holds with $u = 1$ and $v = 0$. The square of a rotation matrix is not particularly interesting. If

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates counterclockwise by θ, then T^2 rotates counterclockwise by 2θ. The trace of T is $\cos\theta + \cos\theta = 2\cos\theta$ and the determinant of T is 1, so $u = 2\cos\theta$ and $v = -1$ and we have $T^2 = (2\cos\theta)\,T - I$, or in other words

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} = (2\cos\theta) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The trigonometric identities that this implies are $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Scores: The median was 93, and the mean was 91.625.

Score	Frequency	Score	Frequency	Score	Frequency
100	1	95	2	88	1
99	1	94	1	87	2
98	2	93	3	86	1
97	2	92	1	84	1
96	2	90	2	67	1