Math 205A Test 1 (50 points)

Name: Solutions

- Check that you have 6 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (10 points) Let $A = \begin{bmatrix} -3 & 2 \\ 2 & -\frac{4}{3} \end{bmatrix}$.

(a) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form.

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 2 & -4/3 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

General solution $\begin{cases} x_1 = \frac{2}{3} x_2 \\ x_2 \text{ free.} \end{cases}$

Parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

(b) Is the vector $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ in Nul A? Explain.

$$A\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -4/3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the vector $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is in Nul A

(c) Do the columns of A span $\mathbb{R}^2$? Explain.

$$\begin{bmatrix} -3 & 2 \\ 2 & -4/3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$$ (as seen in part (a)).

Since we do not have a pivot in every row, the columns of A do not span $\mathbb{R}^2$.

2. (9 points) Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$ be a 5×4 matrix. (The vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$ form the columns of A.) Suppose $2\vec{a}_3 - 5\vec{a}_4 = 3\vec{a}_1 + 2\vec{a}_2$.

(a) Is $\vec{a}_1$ in Span$\{\vec{a}_2, \vec{a}_3, \vec{a}_4\}$? Explain.

$2\bar{a}_3 - 5\bar{a}_4 = 3\bar{a}_1 + 2\bar{a}_2$ gives $\bar{a}_1 = \dfrac{2\bar{a}_3 - 5\bar{a}_4 - 2\bar{a}_2}{3}$

So $\bar{a}_1$ is in Span$\{\bar{a}_2, \bar{a}_3, \bar{a}_4\}$ as it can be written as a linear combination of the other vectors.

(b) Are the columns of A linearly independent? Explain.

As seen in part (a), $\bar{a}_1$ can be written as a linear combination of the other vectors. So the columns of A form a linearly dependent set and hence they are not linearly independent.

(c) Find a non-zero solution of the equation $A\vec{x} = \vec{0}$.

We have $3\bar{a}_1 + 2\bar{a}_2 - 2\bar{a}_3 + 5\bar{a}_4 = \bar{0}$

ie $[\bar{a}_1\ \bar{a}_2\ \bar{a}_3\ \bar{a}_4]\begin{bmatrix} 3 \\ 2 \\ -2 \\ 5 \end{bmatrix} = \bar{0}$. So $\bar{x} = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 5 \end{bmatrix}$ is a non-zero solution of the eqn. $A\bar{x} = \bar{0}$

3. (8 points) Let $H = \left\{ \begin{bmatrix} b \\ a \\ 2a - b \end{bmatrix} : a, b \text{ are real numbers.} \right\}$.

(a) Is H a subspace of $\mathbb{R}^3$? Explain.

$$\begin{bmatrix} b \\ a \\ 2a-b \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Thus, every vector in H can be written as a linear combination of the vectors $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

So $H = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Hence H is a subspace of $\mathbb{R}^3$, since the space spanned by a set of vectors satisfies the conditions in the definition of subspace.

(b) Give a geometric description of H.

$H = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \text{Span}\{\bar{v}_1, \bar{v}_2\}$.

The vectors $\bar{v}_1$ and $\bar{v}_2$ are not multiples of each other (because they have a 0 entry in different places.) Hence, they span a plane. So H is a plane that passes through the origin and contains the vectors $\bar{v}_1, \bar{v}_2$.

4. (10 points) Let $A = \begin{bmatrix} 2 & 3 & -4 \\ -5 & -7.5 & 10 \end{bmatrix}$ and let T be the linear transformation given by $T(\vec{x}) = A\vec{x}$.

(a) What is the codomain of T? Circle only one choice: $\mathbb{R}^2$ $\mathbb{R}^3$

(b) Describe the set of all vectors $\vec{b}$ that are in the range of T. (Show all your calculations by hand.)

Let $\bar{b}$ be in the range of T. Then there is a vector $\bar{x}$ in $\mathbb{R}^3$ such that $T(\bar{x}) = \bar{b}$ i.e. $A\bar{x} = \bar{b}$.

$$\begin{bmatrix} 2 & 3 & -4 & b_1 \\ -5 & -7.5 & 10 & b_2 \end{bmatrix} \underset{R2 = R2 + \frac{5}{2}R1}{\sim} \begin{bmatrix} 2 & 3 & -4 & b_1 \\ 0 & 0 & 0 & b_2 + \frac{5}{2}b_1 \end{bmatrix}$$

So the equation ~~has~~ $A\bar{x} = \bar{b}$ has a solution iff $b_2 + \frac{5}{2}b_1 = 0$.

Thus, the vectors $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in the range must satisfy the condition $b_2 + \frac{5}{2}b_1 = 0$

(c) Is T onto? Explain.

$$A \sim \begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since we do not have a pivot in every row, columns of A do not span $\mathbb{R}^2$.

Hence, T is not onto.

5. (8 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1, x_2, x_3) = (2x_2 - x_3, x_1 + x_2, x_1 + 5x_3).$$

(a) Find the standard matrix of T.

$$\begin{bmatrix} 2x_2 - x_3 \\ x_1 + x_2 \\ x_1 + 5x_3 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

So the standard matrix of $T = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix} = A$ (say).

(b) Is T invertible? Explain. If T is invertible, find a formula for the inverse of T.

$A \sim I_3$. So A is invertible. Hence T is invertible.

$$[A \ \ I_3] \sim \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \ \underbrace{\begin{array}{ccc} -0.556 & 1.111 & -0.111 \\ 0.556 & -0.111 & 0.111 \\ 0.111 & -0.222 & 0.222 \end{array}}_{A^{-1}}\right]$$

Define $S : \mathbb{R}^3 \to \mathbb{R}^3$ by $S(\bar{x}) = A^{-1}\bar{x}$.

Then S is the inverse of T.

$$\text{So } S(\bar{x}) = \begin{bmatrix} -0.556x_1 + 1.111x_2 - 0.111x_3 \\ 0.556x_1 - 0.111x_2 + 0.111x_3 \\ 0.111x_1 - 0.222x_2 + 0.222x_3 \end{bmatrix}$$

6. (5 points) Suppose A is an invertible $n \times n$ matrix such that $A^{-1} = A^T$. Then for any $n \times n$ matrix B, show that $(AB)^T A = B^T$.

$$(AB)^T A = (B^T A^T) A = B^T (A^T A)$$
$$= B^T I_n \quad (\text{since } A^{-1} = A^T)$$
$$= B^T.$$

Thus, $(AB)^T A = B^T$.