Answer Key for Exam #1

1. The lengths of $\begin{pmatrix}1\\1\\1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\\2\\1\end{pmatrix}$ are $\sqrt{1^2+1^2+1^2+0^2+1^2}=\sqrt{4}=2$ and $\sqrt{1^2+1^2+1^2+2^2+1^2}=\sqrt{8}=2\sqrt{2}$ respectively. The dot product of the vectors is $1+1+1+0+1=4$, so the angle θ between them satisfies $\cos\theta=\dfrac{4}{2\cdot 2\sqrt{2}}=\dfrac{1}{\sqrt{2}}$. Therefore $\theta=\frac{\pi}{4}$, or 45°.

2. We use elimination on an augmented matrix. First two steps: subtract the first row from the second, and three times the first row from the third. Next two steps: multiply the second row by -1, and then add twice the second row to the third. This gives

$$\begin{pmatrix}1&4&5&3\\1&2&8&-5\\3&8&2&-45\end{pmatrix}\longrightarrow\begin{pmatrix}1&4&5&3\\0&-2&3&-8\\0&-4&-13&-54\end{pmatrix}\longrightarrow\begin{pmatrix}1&4&5&3\\0&2&-3&8\\0&0&-19&-38\end{pmatrix}.$$

The third equation gives $x_3=2$. Then the second equation becomes $2x_2-6=8$ or $x_2-3=4$, so $x_2=7$. Finally the first equation becomes $x_1+28+10=3$, so $x_1=-35$.

3(a) Step 1: subtract twice the first row of A from the others. Step 2: subtract twice the new second row from the new third row. This gives

$$A=\begin{pmatrix}2&1&2\\4&4&5\\4&6&9\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2\\0&2&1\\0&4&5\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2\\0&2&1\\0&0&3\end{pmatrix}=U.$$

Step 1 puts 2's in the second and third rows of column 1 of L, and step 2 puts a 2 in the third row of column 2 of L, so $L=\begin{pmatrix}1&0&0\\2&1&0\\2&2&1\end{pmatrix}$ and $A=LU$.

3(b) We may now solve $A\vec{x}=\begin{pmatrix}6\\18\\18\end{pmatrix}$ in two steps. First we solve $L\vec{y}=\vec{b}$, which in this case is

$$\begin{pmatrix}1&0&0\\2&1&0\\2&2&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}6\\18\\18\end{pmatrix},$$

and we get $y_1=6$; $12+y_2=18$, so $y_2=6$; and $12+12+y_3=18$, so $y_3=-6$. Then we solve $U\vec{x}=\vec{y}$, which in this case is

$$\begin{pmatrix}2&1&2\\0&2&1\\0&0&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}6\\6\\-6\end{pmatrix}.$$

Here we have $x_3=-2$; $2x_2-2=6$ so $x_2=4$; and $2x_1+4-4=6$ so $x_1=3$.

3(c) We have to reduce $[L\ \ I]$ to $\left[I\ \ L^{-1}\right]$ by row operations, which is not too hard:

$$\begin{pmatrix}1&0&0&1&0&0\\2&1&0&0&1&0\\2&2&1&0&0&1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&1&0&0\\0&1&0&-2&1&0\\0&2&1&-2&0&1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&1&0&0\\0&1&0&-2&1&0\\0&0&1&2&-2&1\end{pmatrix}.$$

Finding U^{-1} is a little more work because not all the pivots are 1. I'll try to avoid fractions:

$$\left(\begin{array}{ccc@{\qquad}ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc@{\qquad}ccc} 6 & 3 & 6 & 3 & 0 & 0 \\ 0 & 6 & 3 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc@{\qquad}ccc} 6 & 3 & 0 & 3 & 0 & -2 \\ 0 & 6 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc@{\qquad}ccc} 12 & 6 & 0 & 6 & 0 & -4 \\ 0 & 6 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc@{\qquad}ccc} 12 & 0 & 0 & 6 & -3 & -3 \\ 0 & 6 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc@{\qquad}ccc} 12 & 0 & 0 & 6 & -3 & -3 \\ 0 & 12 & 0 & 0 & 6 & -2 \\ 0 & 0 & 12 & 0 & 0 & 4 \end{array}\right).$$

Therefore $$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \quad \text{and} \quad U^{-1} = \frac{1}{12}\begin{pmatrix} 6 & -3 & -3 \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}.$$

3(d) Since $A = LU$ and L and U are both invertible, we have $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$. Therefore

$$A^{-1} = \frac{1}{12}\begin{pmatrix} 6 & -3 & -3 \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 6 & 3 & -3 \\ -16 & 10 & -2 \\ 8 & -8 & 4 \end{pmatrix}.$$

4. We have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^{-1} = \frac{1}{5\cdot 8 - 6\cdot 7}\begin{pmatrix} 8 & -6 \\ -7 & 5 \end{pmatrix},$$

so you might guess that $A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -8 & 6 \\ 0 & 0 & 7 & -5 \end{pmatrix}$, and once guessed this is easy to check:

$$\frac{1}{2}\begin{pmatrix} -4 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -8 & 6 \\ 0 & 0 & 7 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = I.$$

Alternatively we can use row operations to reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. I'll try to avoid fractions: subtract three times the first row from the second, and multiply the fourth row by 5 to get

$$\left(\begin{array}{cccc@{\qquad}cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc@{\qquad}cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 35 & 40 & 0 & 0 & 0 & 5 \end{array}\right).$$

Next add the second row to the first, and subtract 7 times the third row from the fourth:

$$\left(\begin{array}{cccc@{\qquad}cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 35 & 40 & 0 & 0 & 0 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cccc@{\qquad}cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -7 & 5 \end{array}\right).$$

The last key step is to add three times the fourth row to the third. Let's also multiply the first row by 2 and the second row by -1:

$$\left(\begin{array}{cccc@{\qquad}cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -7 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cccc@{\qquad}cccc} 2 & 0 & 0 & 0 & -4 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & -20 & 15 \\ 0 & 0 & 0 & -2 & 0 & 0 & -7 & 5 \end{array}\right).$$

Finally, multiply the fourth row by -1 and the third row by $\frac{2}{5}$:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & & -4 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & & 3 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 & & 0 & 0 & -20 & 15 \\ 0 & 0 & 0 & -2 & & 0 & 0 & -7 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & & -4 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & & 3 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & & 0 & 0 & -8 & 6 \\ 0 & 0 & 0 & 2 & & 0 & 0 & 7 & -5 \end{pmatrix}.$$

Dividing all entries by 2 we get A^{-1} as before.

5. From the project we know that

$$R(a)R(b) = \begin{pmatrix} \cos 2a & \sin 2a \\ \sin 2a & -\cos 2a \end{pmatrix}\begin{pmatrix} \cos 2b & \sin 2b \\ \sin 2b & -\cos 2b \end{pmatrix} = \begin{pmatrix} \cos(2a-2b) & -\sin(2a-2b) \\ \sin(2a-2b) & \cos(2a-2b) \end{pmatrix}.$$

Therefore

$$\begin{aligned} R(a)R(b)R(c) &= \begin{pmatrix} \cos(2a-2b) & -\sin(2a-2b) \\ \sin(2a-2b) & \cos(2a-2b) \end{pmatrix}\begin{pmatrix} \cos 2c & \sin 2c \\ \sin 2c & -\cos 2c \end{pmatrix} \\ &= \begin{pmatrix} \cos(2a-2b)\cos 2c - \sin(2a-2b)\sin 2c & \cos(2a-2b)\sin 2c + \sin(2a-2b)\cos 2c \\ \sin(2a-2b)\cos 2c + \cos(2a-2b)\sin 2c & \sin(2a-2b)\sin 2c - \cos(2a-2b)\cos 2c \end{pmatrix} \\ &= \begin{pmatrix} \cos(2a-2b+2c) & \sin(2a-2b+2c) \\ \sin(2a-2b+2c) & -\cos(2a-2b+2c) \end{pmatrix}. \end{aligned}$$

Unlike the product of two reflections, the product of three reflections *is* another reflection, in the line making angle $a-b+c$ with the positive x-axis. The product of an odd number of reflections is a reflection, and the product of an even number of reflections is a rotation.

There are six possible orders for multiplying these matrices in, the others being $R(a)R(c)R(b)$, $R(b)R(a)R(c)$, $R(b)R(c)R(a)$, $R(c)R(a)R(b)$ and $R(c)R(b)R(a)$. (There are six permutations of three things, as we saw with 3×3 permutation matrices.) Since the answer we got above is symmetric in a and c, only three of these six are fundamentally different. $R(c)R(b)R(a)$ is a reflection in the line making angle $c-b+a$ with the positive x-axis, so it must be the same as $R(a)R(b)R(c)$. $R(a)R(c)R(b)$ and $R(b)R(c)R(a)$ are both the reflection in the line making angle $a+b-c$ with the positive x-axis, and $R(b)R(a)R(c)$ and $R(c)R(a)R(b)$ are both the reflection in the line making angle $b+c-a$ with the positive x-axis.

6(a) If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, let's first see how A^2 can equal $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We have

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives us the four equations

$$a^2+bc = 0 = d^2+bc \quad \text{(i)} \qquad \text{and} \qquad b(a+d) = 0 = c(a+d) = 0 \quad \text{(ii)}$$

For (ii) to hold we must have either $a+d=0$ or both $b=0$ and $c=0$. If b and c are both zero then (i) implies $a^2 = 0 = d^2$, so that a and d are both zero also, in which case $a+d=0$. So $a+d$ (which is the trace of A) has to be zero in any case. Note that if $a+d=0$ then $d=-a$ and this means the two equations in (i) are really the same. Note also that the determinant of A is $ad-bc$, and if $a+d=0$ then this becomes $-(a^2+bc)$, which by assumption is zero. We can conclude from all this that if A is a 2×2 matrix, then $A^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if and only if the trace and determinant of A are both zero.

Now we're ready to look at

$$A^3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} a^3+bc(2a+d) & b(a^2+ad+d^2+bc) \\ c(a^2+ad+d^2+bc) & d^3+bc(a+2d) \end{pmatrix}.$$

If this is going to equal $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ then we must have

$$\text{(iii)} \qquad a^3 + bc(2a+d) = 0 \qquad d^3 + bc(a+2d) = 0$$

$$\text{(iv)} \qquad b(a^2 + ad + d^2 + bc) = 0 \qquad c(a^2 + ad + d^2 + bc) = 0$$

The analysis of these is similar to that of (i) and (ii), but somewhat harder. If (iv) holds we must have either $a^2 + ad + d^2 + bc = 0$ or both $b = 0$ and $c = 0$. If b and c are both zero then (iii) implies $a^3 = 0 = d^3$, so that a and d are both zero also. In this case we certainly have $A^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ since we have $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The harder case is $a^2 + ad + d^2 + bc = 0$. We could rewrite this as $a^2 + 2ad + d^2 = ad - bc$, when it says that the determinant is the square of the trace. But we could also rewrite it as $-bc = a^2 + ad + d^2$ and substitute it into one of the equations in (iii). This gives

$$a^3 = -bc(2a+d) = (a^2 + ad + d^2)(2a+d) = 2a^3 + 3a^2d + 3ad^2 + d^3, \quad \text{or} \quad a^3 + 3a^2d + 3ad^2 + d^3 = 0.$$

But this is equivalent to $(a+d)^3 = 0$, which can only mean that $a + d = 0$. So the trace of A must be zero, and above we had that the determinant was the square of the trace, so it must be zero too. The discussion of A^2 then implies that $A^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Alternatively, once we know that $a + d = 0$ then we can simplify the equation $a^2 + ad + d^2 + bc$ by using $d = -a$; it becomes $a^2 - a^2 + a^2 + bc = 0$, which is just $a^2 + bc = 0$. Since $d = -a$ we also have $d^2 + bc = 0$, so both of the equations in (i) are satisfied, as are both of the equations in (ii) since $a + d = 0$. Thus we have again that $A^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

There are a couple of other ways to do this which involve less brute force. If $A^3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ then A can't be invertible—if it was we could hit this equation with A^{-1} three times and get $I = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which is a contradiction. If A is 2×2 and not invertible then the second row of A must be a multiple of the first, so $A = \left(\begin{smallmatrix} ac & ad \\ bc & bd \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\left(\begin{smallmatrix} c & d \end{smallmatrix}\right)$ for some a, b, c, d. Then

$$A^2 = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}(ac+bd)\begin{pmatrix} c & d \end{pmatrix} = (ac+bd)A.$$

Therefore $A^3 = (ac+bd)A^2 = (ac+bd)^2A$. Now if $A^3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ then either $ac + bd = 0$ or $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and in either case we have $A^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The key equation $A^2 = (ac+bd)A$ could also have been obtained from something we proved in class once: if A is 2×2, then $A^2 = (\operatorname{tr} A)A - (\det A)I$, where $\det A$ denotes the determinant of A (zero in this case) and $\operatorname{tr} A$ denotes the trace of A (which is $ac + bd$ in this case).

6(b) There are lots of 3×3 matrices A that work. To construct one large family of examples consider $A = \left(\begin{smallmatrix} 0 & u & 0 \\ v & 0 & w \\ 0 & x & 0 \end{smallmatrix}\right)$. Then

$$A^2 = \begin{pmatrix} uv & 0 & uw \\ 0 & uv + wx & 0 \\ vx & 0 & wx \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 0 & u^2v + uwx & 0 \\ v(uv+wx) & 0 & w(uv+wx) \\ 0 & uvx + wx^2 & 0 \end{pmatrix} = (uv+wx)A.$$

If u, v, w, x are any four numbers such that $uv + wx = 0$ but at least one of uv, uw, vx, wx is not zero, then

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq A^2.$$

Perhaps the simplest example is $u = 1 = w$, $v = 0 = x$, in which case

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$