Exam #1, Math 205B (Linear Algebra)

This take-home exam is due at class time on **Monday, October 6**. (Sooner is fine.) You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the bottom of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Please show all work (though you are encouraged to *check* your answers on MATLAB or a calculator).

1. (12 points) Find the angle between the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

2. (20 points) Solve the system $\begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 8 \\ 3 & 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -45 \end{pmatrix}$ by any method (by hand).

3. (24 points) (a) Find the LU factorization of $A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 4 & 5 \\ 4 & 6 & 9 \end{pmatrix}$.

(b) Use your answer to (a) to solve $A\vec{x} = \begin{pmatrix} 6 \\ 18 \\ 18 \end{pmatrix}$.

(c) Find L^{-1} and U^{-1}.

(d) Use your answers to (c) to compute A^{-1}.

4. (12 points) If $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix}$, find A^{-1}. If you guess the answer, show a check that it works, and explain why you guessed what you did.

5. (20 points) Let $R(\phi)$ be the 2×2 matrix that reflects vectors in $\mathbb{R}^2$ in the line making angle ϕ with the positive x-axis. On the first project you looked at the product of two reflections. This time I want you to try the product of *three* reflections, say

$$R(a)R(b)R(c) = \begin{pmatrix} \cos 2a & \sin 2a \\ \sin 2a & -\cos 2a \end{pmatrix} \begin{pmatrix} \cos 2b & \sin 2b \\ \sin 2b & -\cos 2b \end{pmatrix} \begin{pmatrix} \cos 2c & \sin 2c \\ \sin 2c & -\cos 2c \end{pmatrix}.$$

Is this another reflection? If so, in what line? If not, what is it? How many different orders can you multiply these matrices together in, and how many of the products are genuinely different? (Hint: you should only have to multiply them all together once. You can then get the other products just by switching a, b, c around.)

6. (12 points) Let O_n denote the $n \times n$ zero matrix, in which every entry is zero.

(a) Suppose that A is a 2×2 matrix such that $A^3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = O_2$. Prove that A^2 must also equal O_2.

(b) If A is an $n \times n$ matrix with $n > 2$, then it can happen that $A^3 = O_n$ but $A^2 \neq O_n$. Find an example. (Note: this is probably easier than part (a), so please try it even if you got stuck there.)

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________