Math 206 — First Midterm

October 5, 2012

Name: EXAM SOLUTIONS

Instructor: __________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 8 pages including this cover AND IS DOUBLE SIDED. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.
5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.
6. You may use any previously permitted calculator. However, you must state when you use it.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones and hats.
9. Remember that this is a chance to show what you've learned, and that the questions are just prompts.

Problem	Points	Score
1	18	
2	20	
3	22	
4	14	
5	14	
6	12	
Total	100	

1. [18 points]

a. [6 points] Describe and sketch the surface defined by $z + 88 = -9x^2 + 54x - 4y^2 - 16y$.

Solution: This is a complete the square problem. We can simplify as $z + 88 = -9(x - 3)^2 + 81 - 4(y + 2)^2 + 16$. This is the equation for a downward facing parabolic bowl.

b. [6 points] Write down the parametrization $r(t)$ for the intersection of this surface with the surface $z = x^2$.

Solution: This is done by substituting x^2 for z and solving again

$$88 = -10x^2 + 54x - 4y^2 - 16y.$$

You complete the square in the same way as above and get the ellipse $88 = -10(x - 2.7)^2 + 72.9 - 4(y + 2) + 16$. This is $0.9 = 10(x - 2.7)^2 + 4(y + 2)^2$ and we parametrize it by $p(\theta) = (0.3\cos(x) + 2.7, \sqrt{0.9/4}\sin(y) - 2)$.

c. [6 points] Calculate the equation for the tangent plane to the parabolic bowl at $(4, -2, 0)$.

Solution: We need the Jacobian for our equation in part (a). This is $[-18x + 54, -8y - 16]$. Then we can use the formula

$$f(4, 2) + [-18(4) + 54, -8(-2) - 16][x - 4, y + 2] = z$$

which is $0 - 18x + 72 = z$.

2. [20 points]

a. [5 points] Check that the two lines with the following parametrizations intersect at the point $(1, 2, -4)$:

- $r_1(t) = [0, 2, -3] + t[1, 0, -1]$
- $r_2(s) = [-2, 3, -6] + s[3, -1, 2]$.

Solution: We just need to check that the point is on both curves. Note that this works with $r_1(1) = [1, 2, -4]$ and $r_2(1) = [1, 2, -4]$.

b. [7 points] Let v be the vector defined by $r_1(2) - r_1(1)$, and let w be the vector defined by $r_2(2) - r_2(1)$. Calculate the vector a defined as $a = (v \times w)$.

Solution: We have that $v = [1, 0, -1]$ and $w = [3, -1, 2]$. The cross product is $[-1, -5, -1]$

c. [5 points] Let $b = (x, y, z)$ be any vector such that b is perpendicular to a. Solve for the equation defining all such b.

Solution: We need the dot product to be zero, so this is: $a \cdot b = 0$ which is $-x - 5y - z = 0$.

d. [5 points] Graph the equation from part (c), and in the same graph, plot the curves r_1 and r_2.

Solution: It's a line.

3. [22 points] Given a point in rectilinear coordinates (x, y, z) there is a function $f(x, y, z) = (r, \theta, w)$ which gives us the cylindrical coordinates, and another function $g(x, y, z) = (\rho, \theta, \phi)$ which gives us the spherical coordinates. The Martians have a slightly different way of describing vectors (for a full description, read "The Martian Chronicles"). They use (a, b, c) which satisfy $(x, y, z) = M(a, b, c) = (3a\cos(b)\sin(c), 4a\sin(b)\sin(c), 7a\cos(c))$.

a. [8 points] Calculate the Jacobians of f and M.

Solution: We first need to write down f Recall that this is $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ and $w = z$.

$$\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{yx^2}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The jacobian for M is likewise

$$\begin{pmatrix} 3\cos(b)\sin(c) & -3a\sin(b)\sin(c) & 3a\cos(b)\cos(c) \\ 4\sin(b)\sin(c) & 4a\cos(b)\sin(c) & 4a\sin(b)\cos(c) \\ 7\cos(c) & 0 & -7a\sin(c) \end{pmatrix}$$

b. [6 points] Plot the rectilinear coordinate $(1, 1, \sqrt{2})$ and convert to cylindrical and spherical coordinates.

Solution: We use our conversion formulas to get that in cylindrical coordinates this point is $(\sqrt{2}, \pi/4, \sqrt{2})$, and in spherical coordinates it is $(2, \pi/4, \pi/4)$.

c. [8 points] Sketch the Martian equation $a = 2$.

Solution: Note that the formula the Martians use is very similar to our spherical coordinate system. The difference is the 3 the 4 and the 7. All this does is stretch our sphere. This is because Martians prefer ellipsoids (don't believe me? Read the Martian Chronicles–well, it's not explicitly stated there, but it is in the subtext)–the Martian surface is an ellipsoid that goes up to 7 on the z-axis, 3 on the x-axis and 4 on the y-axis.

4. [14 points] Calculate the following limits, or demonstrate that they do not exist:

a. [7 points]

$$f(x,y) = \begin{cases} \frac{y^4}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

What is $\lim_{(x,y)\to(0,0)} f(x,y)$ or does it not exist?

Solution: This is like the homework problem with the squeeze theorem. We have that:

$$0 \leq \frac{y^4}{x^4+y^2} = y^2 \frac{y^2}{x^4+y^2} \leq y^2 \frac{y^2+x^4}{x^4+y^2} = y^2.$$

Since the limit of y^2 as $(x,y) \to (0,0)$ is 0, our original limit is zero.

b. [7 points] Does

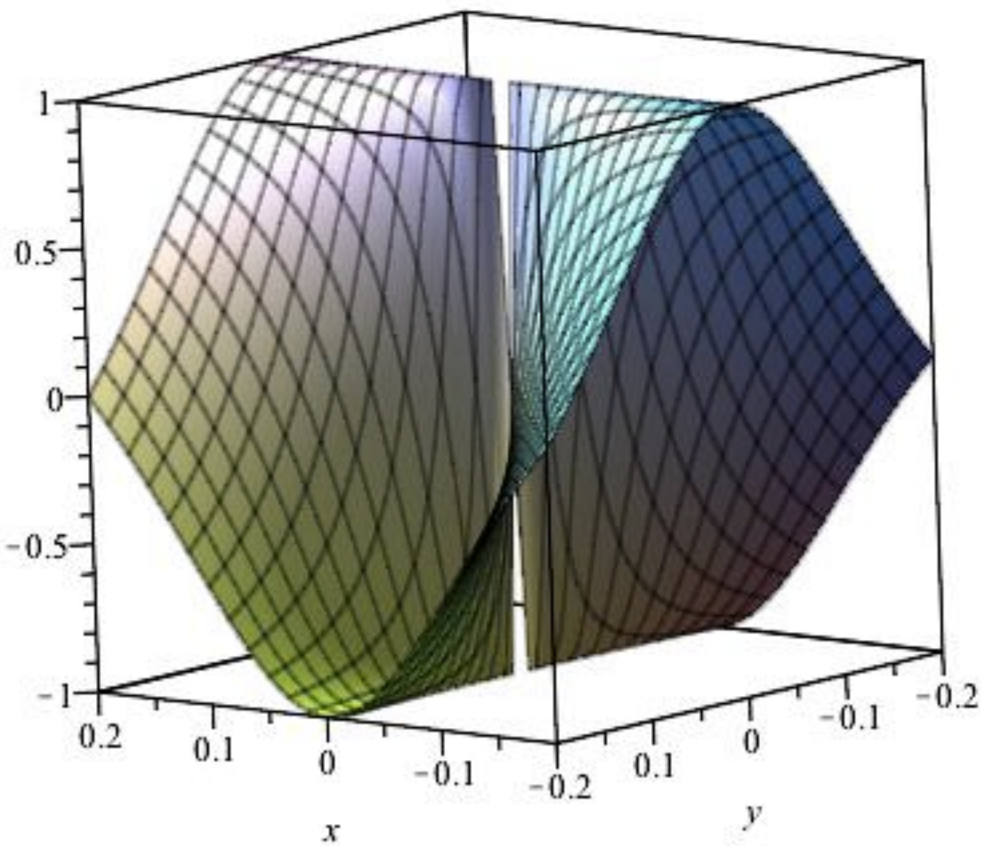
$$\lim_{(x,y)\to(0,0)} \frac{\ln(1-x^2-y^2)+x^2+y^2}{x^2+y^2}$$

exist? If so, what is it?

Solution: We can instead do the limit as $r \to 0$ since that takes care of every direction and approach at once. We get that this limit is equal to

$$\lim_{r\to 0} \frac{\ln(1-r^2)+r^2}{r^2} = 1.$$

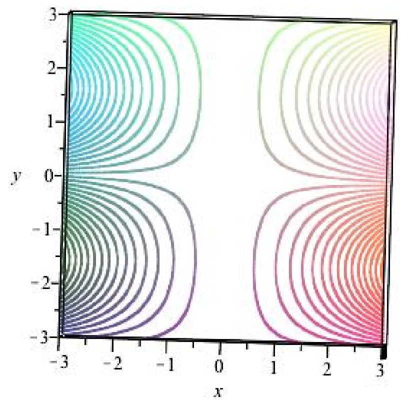
5. [14 points] Here is the graph of a function $f(x, y) = z$:

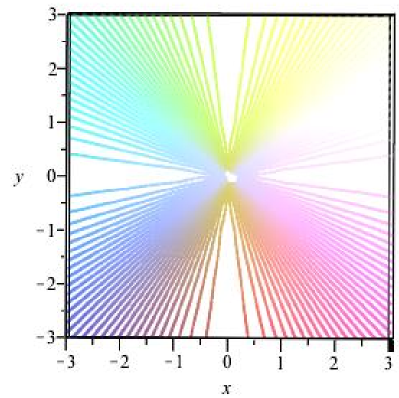


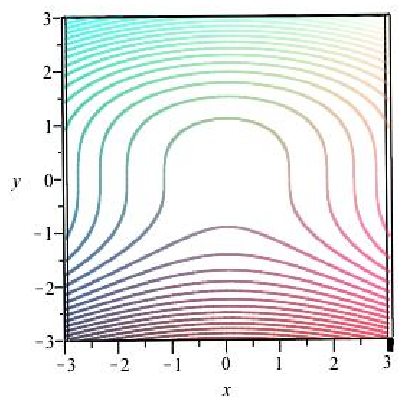
a. [7 points] What is the $\lim_{(x,y)\to(0,0)} f(x, y)$? Explain how you calculated it.

Solution: It does not exist! If you approach along the line $x = 0$ you get one value, and if you approach along the line $y = 0$ you get the negative of that.

b. [7 points] Circle the plot of the level curves of $f(x, y)$. Briefly explain your choice.







6. [12 points]

a. [6 points] Sketch the image of the unit square under $f(x, y) = (x - y^2, y)$.

Solution: We check what happens to the four lines $(x, 0)$ with $0 \leq x \leq 1$ and $(1, y)$ with $0 \leq y \leq 1$ and $(x, 1)$ with $0 \leq x \leq 1$ and $(0, y)$ with $0 \leq y \leq 1$. The first is unchanged. The second becomes $(1 - y^2, y)$ which is the curve $1 - y^2 = x$. The third is $(x - 1, 1)$, and the last is $(-y^2, y)$ which is the curve $-y^2 = x$.

b. [6 points] Sketch the level curves for $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$ with $c = 0, 1$.

Solution: We set the function equal to zero and get that the only solution is $x = y = 0$, so that is the level curve.
We set the function equal to one and get the ellipse as the level curve with $c = 1$.