Math 205 A B Fall 2012 NAME (legibly!) suggested solutions.

EXAM 1 October 5, 2012 Circle Your SECTION: A (8 am) B (9:30 am)

DO NOT WRITE HERE!

1	
2	
3	
4	
5	
6	
7	
TOTAL	

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!

1. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a transformation. Then we say T is a *linear* transformation if T satisfies what two conditions? (Note: in addition to a couple equalities, your conditions will include the words "for all" in appropriate places).

① $T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$.

② $cT(\vec{u}) = T(c\vec{u})$ for all scalars $c \in \mathbb{R}$ and vectors $\vec{u} \in \mathbb{R}^n$.

2. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1x_2 \\ x_2 + 4x_3 \\ x_1x_2x_3 \end{bmatrix}$.

(2A) Use the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and scalar $c = 10$ to illustrate whether T does or does not satisfy the two conditions in problem (1).

① $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 9 \\ 11 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 13 \\ 12 \end{bmatrix} = \begin{bmatrix} 21 \\ 24 \\ 18 \end{bmatrix}$

while $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 60 \\ 24 \\ 100 \end{bmatrix}$. Since $\begin{bmatrix} 21 \\ 24 \\ 18 \end{bmatrix} \neq \begin{bmatrix} 60 \\ 24 \\ 100 \end{bmatrix}$, condition 1 is not satisfied

② $cT(\vec{u}) = 10\begin{bmatrix} 9 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 90 \\ 110 \\ 60 \end{bmatrix}$

while $T(c\vec{u}) = T\left(10\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}\right) = \begin{bmatrix} 900 \\ 110 \\ 6000 \end{bmatrix}$

Since $\begin{bmatrix} 90 \\ 110 \\ 60 \end{bmatrix} \neq \begin{bmatrix} 900 \\ 110 \\ 6000 \end{bmatrix}$, T fails to satisfy the second condition.

(or you could also show that $10\,T(\vec{v}) \neq T(10\vec{v})$)

(2B) Do your results in 2A say T is *not* a linear transformation or do they support the conclusion that T *is* a linear transformation?

Because T doesn't satisfy either condition, T is not a linear transformation.

((and in fact, because it fails even one condition, T is NOT a L.T.))

3. Suppose that T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix $\begin{bmatrix} 10 & 8 & 0 \\ 7 & 5 & 3 \\ 5 & 4 & 0 \\ 3 & 6 & -18 \end{bmatrix}$.

(3A) What are the domain and codomain, respectively, for this T?

The domain is … $\mathbb{R}^3$ The codomain is … $\mathbb{R}^4$

(3B) Find the image under T of the vector $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right) = 3\begin{bmatrix} 10 \\ 7 \\ 5 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 8 \\ 5 \\ 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 0 \\ -18 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ 15 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 0 \\ -36 \end{bmatrix} = \begin{bmatrix} 30 \\ 27 \\ 15 \\ -27 \end{bmatrix}$$

(3C) Determine if the vector $\mathbf{u} = \begin{bmatrix} 9 \\ 8 \\ 4 \\ -6 \end{bmatrix}$ is in the range of T. If it is, find in parametric vector form all $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{u}$. But if $\mathbf{u}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

the vector $\vec{u}$ will be in the range of $T \Leftrightarrow A\vec{x} = \vec{u}$ has a soln.

The RREF of $A|\vec{u}$ is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, and in particular,

the 3rd row represents the eqn $0x_1 + 0x_2 + 0x_3 = 1$. So $A\vec{x} = \vec{u}$ is an inconsistent system, i.e. it has no soln, and thus, $\vec{u} \notin$ range of T.

(3C) Determine if the vector $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \\ 4 \\ -6 \end{bmatrix}$ is in the range of T. If it is, find in parametric vector form all $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{v}$. But if $\mathbf{v}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

Here RREF of $(A|\vec{v})$ is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2\frac{2}{3} \\ 0 & 1 & -5 & -2\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. the system $A\vec{x} = \vec{v}$ is consistent, and there's a free variable.

We have that $T\vec{x} = \vec{v}$ where

$$\vec{x} = \begin{bmatrix} 2\frac{2}{3} \\ -2\frac{1}{3} \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$ where x_3 is free.

4. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors that belong to $\mathbb{R}^j$ for some j. Give the correct definition of what it means to say S is a linearly independent set.

The set S is a linearly independent set $\Leftrightarrow$ the only solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_K\vec{v}_K = \vec{0}$ is the trivial solution $x_1 = x_2 = \cdots = x_K = 0$.

5. In each part below, find a set S of vectors in $\mathbb{R}^3$ that satisfy the condition(s), or explain why there is no such set. (Three separate problems)

(5A) The set S is linearly independent, and contains exactly two different non-zero vectors.

one example $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(5B) The set S is linearly dependent, and contains exactly three different non-zero vectors.

a common example $S = \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ (Check the RREF! or note that $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$)

an easy example: $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

(5C) The set S has five vectors and is linearly independent.

This cannot occur. If $S = \{\vec{v}_1, \ldots, \vec{v}_5\}$ and each vector $\vec{v}_i \in \mathbb{R}^3$ then the corresponding matrix equation $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix} \vec{x} = \vec{0}$ will have at <u>least</u> 2 free variables, since the matrix is 3×5, which means the equation $x_1\vec{v}_1 + \cdots + x_5\vec{v}_5 = \vec{0}$ will have (infinitely many) <u>non trivial</u> sol'n's.

6. Let $M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix}$. Let $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5\}$ be the set of column vectors of M.

Fact: the RREF of $\left[\begin{array}{ccccc|cccc} 1 & 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 3 & 0 & 1 & 0 & 0 \\ 5 & 2 & 10 & 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 2 & 13 & -7 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 6 & -5 & 2 \\ 0 & 1 & 0 & 16 & 7 & 0 & -25 & 23 & -10 \\ 0 & 0 & 1 & -3 & -1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13 & -12 & 5 \end{array}\right]$.

(6A) Use the *fact* to find conditions on $b_1, \ldots, b_4$ which guarantee that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is in the span of the set S.

The system of equations represented by $M\vec{x} = \vec{b}$ will be consistent if and only if $\boxed{0 = b_1 + 13b_2 - 12b_3 + 5b_4}$

(6B) We know that each member of S is in the span of S, and therefore the components of each member must satisfy the condition(s) you found in (6A). Indeed show that the components of $\mathbf{c}_4$ satisfy your condition(s):

$\vec{c}_4 = \begin{bmatrix} 7 \\ 4 \\ 2 \\ -7 \end{bmatrix}$; does $0 = 7 + 13\cdot 4 - 12\cdot 2 + 5\cdot(-7)$?

RHS ↗ is $7 + 52 - 24 - 35$

$= 59 - 59$

$= 0$ ✓ yes.

(6C) Use your condition(s) from 6A to find the value of b_3 for which $\mathbf{b} = \begin{bmatrix} 11 \\ 3 \\ b_3 \\ 2 \end{bmatrix}$ is in the span of the set S.

we need $0 = 11 + 13\cdot 3 - 12\cdot b_3 + 5\cdot 2$

$\therefore 12b_3 = 11 + 39 + 10 = 60$

$b_3 = \frac{60}{12} = 5$

7. This problem uses the same M and S as the previous problem, and the information is copied here:

Let $M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix}$. Let $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5\}$ be the set of column vectors of M.

Fact: the RREF of $\left[\begin{array}{ccccc|cccc} 1 & 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 3 & 0 & 1 & 0 & 0 \\ 5 & 2 & 10 & 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 2 & 13 & -7 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 6 & -5 & 2 \\ 0 & 1 & 0 & 16 & 7 & 0 & -25 & 23 & -10 \\ 0 & 0 & 1 & -3 & -1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13 & -12 & 5 \end{array}\right]$.

7A) Which members of S can be expressed as linear combinations of the *other* vectors in S?

The RREF tells us that $\vec{x}$ is a sol'n of $M\vec{x} = \vec{0} \Leftrightarrow \vec{x} = x_4 \begin{bmatrix} 0 \\ -16 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -7 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ where x_4 & x_5 are free.

Thus in $\overbrace{x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_5\vec{c}_5 = \vec{0}}^{\circledast}$, it's possible to choose x_4 and/or x_5 in ways that cause x_2 x_3 x_4 and x_5 to be nonzero, and thus when $x_i \neq 0$ the corresponding eq'n can be solved for $\vec{c}_i$, $\underline{\text{for } i = 2, 3, 4, 5.}$

But since $x_1 = 0$ in $\underline{\text{any}}$ sol'n of $\circledast$, it's impossible to write $\vec{c}_1$ as a L.C. of the other vectors in S. So the answer to the question is, $\boxed{\vec{c}_2, \vec{c}_3, \vec{c}_4 \text{ and } \vec{c}_5}$

7B) Find a way to express $\mathbf{c}_4$ as a linear combination of $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_5$, without using weights of 0, or explain why this is impossible.

Let's begin by finding a sol'n of $\circledast$ in which none of x_4 x_2 x_3 and x_5 are 0.

Hopefully choosing $x_4 = x_5 = 1$ will do (at least this choice makes $\underline{\text{two}}$ of the weights nonzero!)

so: let $x_4 = x_5 = 1$. Then

$x_2 = -16x_4 - 7x_5 = -16 - 7 = -23$

and $x_3 = 3x_4 + x_5 = 3 + 1 = 4$.

Thus $-23\vec{c}_2 + 4\vec{c}_3 + \vec{c}_4 + \vec{c}_5 = \vec{0}$.

solving for $\vec{c}_4$ we find

$$\boxed{\vec{c}_4 = 23\vec{c}_2 - 4\vec{c}_3 - \vec{c}_5}$$

(of course there are ∞-many ways to answer this question.)