1. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a transformation. Then we say T is a *linear* transformation if T satisfies what two conditions? (Note: in addition to a couple equalities, your conditions will include the words "for all" in appropriate places).

2. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3\,x_1\,x_2 \\ x_2 + 4x_3 \\ x_1\,x_2\,x_3 \end{bmatrix}$.

(2A) Use the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and scalar $c = 10$ to illustrate whether T does or does not satisfy the two conditions in problem (1).

(2B) Do your results in 2A say T is *not* a linear transformation or do they support the conclusion that T *is* a linear transformation?

3. Suppose that T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix $\begin{bmatrix} 10 & 8 & 0 \\ 7 & 5 & 3 \\ 5 & 4 & 0 \\ 3 & 6 & -18 \end{bmatrix}$.

(3A) What are the domain and codomain, respectively, for this T?

The domain is ...

The codomain is ...

(3B) Find the image under T of the vector $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

(3C) Determine if the vector $\mathbf{u} = \begin{bmatrix} 9 \\ 8 \\ 4 \\ -6 \end{bmatrix}$ is in the range of T. If it is, find in parametric vector form all $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{u}$. But if $\mathbf{u}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

(3D) Determine if the vector $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \\ 4 \\ -6 \end{bmatrix}$ is in the range of T. If it is, find in parametric vector form all $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{v}$. But if $\mathbf{v}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

4. Suppose that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_k\}$ is a set of vectors that belong to $\mathbb{R}^j$ for some j. Give the correct definition of what it means to say S is a linearly independent set.

5. In each part below, find a set S of vectors in $\mathbb{R}^3$ that satisfy the condition(s), or explain why there is no such set. (Three separate problems)

(5A) The set S is linearly independent, and contains exactly two different non-zero vectors.

(5B) The set S is linearly dependent, and contains exactly three different non-zero vectors.

(5C) The set S has five vectors and is linearly independent.

6. Let $M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix}$. Let $S = \{\mathbf{c}_1,\ \mathbf{c}_2,\ \ldots,\ \mathbf{c}_5\}$ be the set of column vectors of M.

Fact: the RREF of $\left[\begin{array}{ccccc|cccc} 1 & 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 3 & 0 & 1 & 0 & 0 \\ 5 & 2 & 10 & 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 2 & 13 & -7 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 6 & -5 & 2 \\ 0 & 1 & 0 & 16 & 7 & 0 & -25 & 23 & -10 \\ 0 & 0 & 1 & -3 & -1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13 & -12 & 5 \end{array}\right]$.

(6A) Use the *fact* to find conditions on $b_1, \ldots, b_4$ which guarantee that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is in the span of the set S.

(6B) We know that each member of S is in the span of S, and therefore the components of each member must satisfy the condition(s) you found in (6A). Indeed show that the components of $\mathbf{c}_4$ satisfy your condition(s):

(6C) Use your condition(s) from 6A to find the value of b_3 for which $\mathbf{b} = \begin{bmatrix} 11 \\ 3 \\ b_3 \\ 2 \end{bmatrix}$ is in the span of the set S.

7. This problem uses the same M and S as the previous problem, and the information is copied here:

Let $M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix}$. Let $S = \{\mathbf{c}_1,\ \mathbf{c}_2,\ \ldots\ ,\ \mathbf{c}_5\}$ be the set of column vectors of M.

Fact: the RREF of $\left[\begin{array}{ccccc|cccc} 1 & 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 3 & 0 & 1 & 0 & 0 \\ 5 & 2 & 10 & 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 2 & 13 & -7 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 6 & -5 & 2 \\ 0 & 1 & 0 & 16 & 7 & 0 & -25 & 23 & -10 \\ 0 & 0 & 1 & -3 & -1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13 & -12 & 5 \end{array}\right]$.

7A) Which members of S can be expressed as linear combinations of the *other* vectors in S?

7B) Find a way to express $\mathbf{c}_4$ as a linear combination of $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_5$, without using weights of 0, or explain why this is impossible.