1. Suppose the augmented matrix of the matrix equation $A\mathbf{x} = \mathbf{b}$ is: $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 3 & 6g \\ 0 & 0 & 0 & -5 & 8k \\ 0 & 0 & 0 & 0 & 10p \end{array}\right]$

1A. This augmented matrix is not quite in reduced row echelon form. Find the reduced row echelon form; *circle your final matrix.*

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 3 & 6g \\ 0 & 0 & 0 & -5 & 8K \\ 0 & 0 & 0 & 0 & 10p \end{array}\right] \underset{\substack{r_3/3 \to r_3 \\ \text{THEN} \\ r_4 + 5r_3 \to r_4}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 1 & 2g \\ 0 & 0 & 0 & 0 & 8K+10g \\ 0 & 0 & 0 & 0 & 10p \end{array}\right] \overset{r_1 - 7r_3 \to r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 2a-14g \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 1 & 2g \\ 0 & 0 & 0 & 0 & 8K+10g \\ 0 & 0 & 0 & 0 & 10p \end{array}\right]$$

note: some people divided row 4 by -5 first and proceeded from there; the final column can take on different forms [until a step like this ⟶ ⟶]

I accepted this ↑ as the final answer. However, if $8K+10g \neq 0$, the final col. becomes $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$! .. and there are other possibilities!

1B. Use the answer to (1A) to answer these questions: What conditions are there on a, c, g, k and p so that $A\mathbf{x} = \mathbf{b}$ has…

1B-i. …NO solutions. (tell me about all of a, c, g, k and p in *each* of these three parts).

if $8K+10g \neq 0$ OR $10p \neq 0$ ["or both" is understood], $a \in \mathbb{R}$, $c \in \mathbb{R}$

[there are similar answers if row reduction was carried out in a different order. But these answers are equivalent]

1B-ii. …exactly one solution.

can't happen. The existence of a free variable implies that there are ∞-many soln's if no inconsistency, and none otherwise.

1B-iii. …infinitely many solutions.

if $8K+10g = 0$ AND $10p = 0$, $a \in \mathbb{R}$, $c \in \mathbb{R}$

1C: Suppose that a, c, g, k and p are in fact chosen so that $A\mathbf{x} = \mathbf{b}$ is consistent. Give the particular solution obtained by setting any and all free variables equal to 2; your answer will be in terms of a, c, g, k and p.

in general, $\begin{cases} x_1 = 2a - 14g - 4x_3 \\ x_2 = 4c - 3x_3 \\ x_3 = \text{free} \\ x_4 = 2g \end{cases}$; taking $x_3 = 2$ yields $\begin{cases} x_1 = 2a - 14g - 8 \\ x_2 = 4c - 6 \\ x_3 = 2 \\ x_4 = 2g \end{cases}$ these could LOOK different but represent equivalent answers if your reduction step seq was different

2. Suppose a linear transformation $T : \mathbf{R}^s \to \mathbf{R}^t$ is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is this matrix:

$$A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$$

2A. What are s and t? $s =$ 4 $t =$ 3

2B. Find $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$. $= 2\begin{bmatrix} 9 \\ 2 \\ -4 \end{bmatrix} + 1\begin{bmatrix} 98 \\ 18 \\ -44 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \\ -8 \end{bmatrix} + \begin{bmatrix} 98 \\ 18 \\ -44 \end{bmatrix} = \begin{bmatrix} 116 \\ 22 \\ -52 \end{bmatrix}$

2C. Is $\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}\right) \in T$? *Explain your answer.* No. Since T is a function and 2B shows that $\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 116 \\ 22 \\ -52 \end{bmatrix}\right) \in T$ there can't be a different ordered pair in T having $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ as its 1st element.

To answer the next questions, the following fact is useful:

$$\left[\begin{array}{cccc|ccc} 9 & -31 & 98 & -84 & 1 & 0 & 0 \\ 2 & -5 & 18 & -13 & 0 & 1 & 0 \\ -4 & 14 & -44 & 38 & 0 & 0 & 1 \end{array}\right] \text{ is row equivalent to } \left[\begin{array}{cccc|ccc} 1 & 0 & 4 & 1 & -5 & 3 & -10 \\ 0 & 1 & -2 & 3 & -2 & 1 & -4 \\ 0 & 0 & 0 & 0 & 8 & -2 & 17 \end{array}\right]$$

2D. What (if any) conditions are there on b_1, b_2 and b_3 so that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the image of T?

$$8b_1 - 2b_2 + 17b_3 = 0$$

2E. Verify that $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ does in fact satisfy your conditions in (2D); show the work.

$8 \cdot 5 - 2 \cdot 3 + 17 \cdot (-2) =$

$40 - 6 - 34 =$

$40 - 40 = 0$ as expected.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 & x_4 are free.

2F. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$; express your answer in parametric ($\mathbf{x} = \mathbf{p} + \mathbf{v}_h$) form.

this is equivalent to solving $A\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. The augment matrices above show if $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ then

$$\begin{cases} x_1 = (-5 \cdot b_1 + 3b_2 - 10b_3) - 4x_3 - x_4 \\ x_2 = (-2b_1 + b_2 - 4b_3) + 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

becomes

$$\begin{cases} x_1 = (-5 \cdot 5 + 3 \cdot 3 - 10 \cdot -2) - 4x_3 - x_4 \\ x_2 = (-2 \cdot 5 + 3 - 4 \cdot -2) + 2x_3 - 3x_4 \\ x_3 = \text{free}, \ x_4 = \text{free} \end{cases}$$

$$\begin{cases} x_1 = 4 - 4x_3 - x_4 \\ x_2 = 1 + 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

3A. What does it mean to say that a set of vectors $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\}$ is *linearly independent*? (Give the definition).

We say a set $S = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_p\}$ is linearly independent

$\Longleftrightarrow$

the only solution to $x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_p\vec{c}_p = \vec{0}$ is $x_1 = x_2 = \cdots = x_p = 0$.

For the rest of this question let $S = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ be the column vectors of $A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$.

3B. Find all solutions of $A\mathbf{x} = \mathbf{0}$. (Note the row reduced version of A appears in question (2)).

$$A \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -4x_3 - x_4 \\ x_2 = 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

(alternative notation $\vec{x} = x_3\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ where x_3 & x_4 are free.)

3C. Explicitly express $\mathbf{0}$ as a linear combination of $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$ such that the weight of $\mathbf{c}_3$ is one, or explain why this is impossible.

Since x_3 is free, we can choose it - the weight of $\vec{c}_3$ - to be 1.
Since x_4 is free, we can choose it to be 0 since we don't care if $\vec{c}_4$ appears in our L.C. Then $x_1 = -4 \cdot 1 - 0 = -4$ AND $x_2 = 2 \cdot 1 - 0 = 2$ so:

$$-4\vec{c}_1 + 2\vec{c}_2 + \vec{c}_3 = \vec{0}$$

(there are ∞-many answers of course! (by choosing x_4 differently))

3D. Express $\mathbf{c}_3$ as a linear combination of the other columns, or explain why this cannot be done.

Solving 3C's answer for $\vec{c}_3$ gives

$$\vec{c}_3 = 4\vec{c}_1 - 2\vec{c}_2$$

3E. Is S linearly independent or linearly dependent? *Explain your answer* in terms of the definitions or any theorems we stated in class.

S is Linearly Dependent. ① At least one vector in S is a L.C. of the others, as shown in 3D.

② the matrix A of column vectors from S has more columns than rows

③ the set is not L.I. because 3C shows there are L.C.'s of $\vec{c}_1, \ldots, \vec{c}_4$ adding to $\vec{0}$ yet not all weights are 0

— etc —

4. Define $T : X \to \mathbf{R}^3$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-ab} \\ 0 \\ 2a+3b \end{bmatrix}$, where X is the largest subset of $\mathbf{R}^2$ for which all the formulas can be computed.

4A. Find $T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$. $= \begin{bmatrix} \sqrt{9-5} \\ 0 \\ 2\cdot 1 + 3\cdot 5 \end{bmatrix} = \begin{bmatrix} \sqrt{4} \\ 0 \\ 2+15 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 17 \end{bmatrix}$

4B. Explicitly describe X.

$$X = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, 9-ab \geq 0 \right\} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, ab \leq 9 \right\}$$

4C. Find a smaller codomain for T than $\mathbf{R}^3$; explain your answer.

for any ab, $\sqrt{9-ab} \geq 0$; also $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ always has a 0 "in the middle" so a smaller codomain could be $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x \geq 0, y = 0, z \in \mathbb{R} \right\}$. There are other possible answers.

4D. Give an explicit counter example that shows why T is not a linear transformation. Explain what you're doing.

many students used this:

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9} \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} \sqrt{9} \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$$

yet $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-1} \\ 0 \\ 2\cdot 1 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} \sqrt{8} \\ 0 \\ 5 \end{bmatrix}$;

Since $\begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} \neq \begin{bmatrix} \sqrt{8} \\ 0 \\ 5 \end{bmatrix}$ T is not a L.T. (others found examples where $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$ could be computed but $T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$ could not; this was fine.

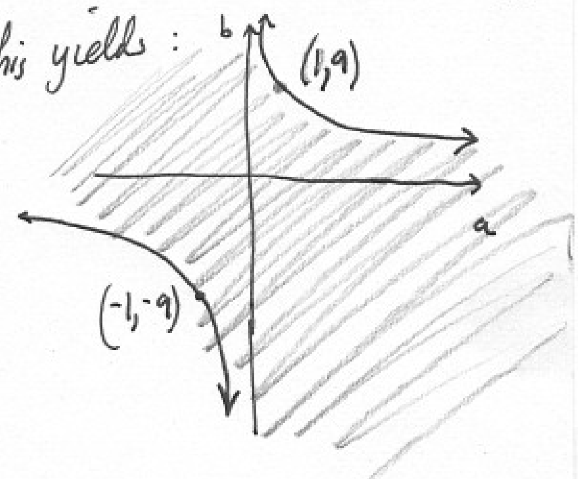
4E. *Bonus!* Sketch the domain of T.

you need to plot "$b \leq 9/a$" if $a > 0$

$b =$ any # if $a = 0$

$b \geq 9/a$ if $a < 0$ [inequalities change sign when both sides are divided by a negative]

this yields:



4F. *Bonus!* Sketch the image of T. (This will be tough).

the hard part is in showing that the codomain in 4C is actually the image of T. This is not easy. It involves demonstrating that no matter HOW you pick $p \geq 0$ and r, there is a pair a, b for which $\sqrt{9-ab} = p$ AND $2a + 2b = r$...