

1. Suppose the augmented matrix of the matrix equation $Ax = b$ is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 3 & 6g \\ 0 & 0 & 0 & -5 & 8k \\ 0 & 0 & 0 & 0 & 10p \end{array} \right]$$

1A. This augmented matrix is not quite in reduced row echelon form. Find the reduced row echelon form; *circle your final matrix*.

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 3 & 6g \\ 0 & 0 & 0 & -5 & 8k \\ 0 & 0 & 0 & 0 & 10p \end{array} \right] \xrightarrow{\substack{r_3/3 \rightarrow r_3 \\ \text{THEN} \\ r_4 + 5r_3 \rightarrow r_4}} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 1 & 2g \\ 0 & 0 & 0 & 0 & 8k+10g \\ 0 & 0 & 0 & 0 & 10p \end{array} \right] \xrightarrow{r_1 - 7r_3 \rightarrow r_1} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 2a - 14g \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 1 & 2g \\ 0 & 0 & 0 & 0 & 8k+10g \\ 0 & 0 & 0 & 0 & 10p \end{array} \right]$$

note: some people divided row 4 by -5 first and proceeded from there; the final column can take on different forms [until a step like this → →]

I accepted this as the final answer. However, if $8k+10g \neq 0$, the final col. becomes $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$! ... and there are other possibilities!

1B. Use the answer to (1A) to answer these questions: What conditions are there on a , c , g , k and p so that $Ax = b$ has...

1B-i. ... NO solutions. (tell me about all of a , c , g , k and p in each of these three parts).

if $8k+10g \neq 0$ OR $10p \neq 0$ ["or both" is understood], $a \in \mathbb{R}$, $c \in \mathbb{R}$

[these are similar answers if row reduction was carried out in a different order. BT these answers are equivalent]

1B-ii. ... exactly one solution.

can't happen. The existence of a free variable implies that there are ∞ -many soln's if no inconsistency, and none otherwise.

1B-iii. ... infinitely many solutions.

if $8k+10g = 0$ AND $10p = 0$, $a \in \mathbb{R}$, $c \in \mathbb{R}$

1C: Suppose that a , c , g , k and p are in fact chosen so that $Ax = b$ is consistent. Give the particular solution obtained by setting any and all free variables equal to 2; your answer will be in terms of a , c , g , k and p .

in general,

$$\begin{cases} x_1 = 2a - 14g - 4x_3 \\ x_2 = 4c - 3x_3 \\ x_3 = \text{free} \\ x_4 = 2g \end{cases}$$

; taking $x_3 = 2$ yields

$$\begin{cases} x_1 = 2a - 14g - 8 \\ x_2 = 4c - 6 \\ x_3 = 2 \\ x_4 = 2g \end{cases}$$

these could look different but represent equivalent answers if your reduction steps were different

2. Suppose a linear transformation $T: \mathbb{R}^s \rightarrow \mathbb{R}^t$ is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is this matrix:

2A. What are s and t ? $s = \underline{4}$ $t = \underline{3}$ $A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$

2B. Find $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 9 \\ 2 \\ -4 \end{bmatrix} + 1\begin{bmatrix} 98 \\ 18 \\ -44 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \\ -8 \end{bmatrix} + \begin{bmatrix} 98 \\ 18 \\ -44 \end{bmatrix} = \begin{bmatrix} 116 \\ 22 \\ -52 \end{bmatrix}$

2C. Is $\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}\right) \in T$? Explain your answer. **No.** Since T is a function and 2B shows that $\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 116 \\ 22 \\ -52 \end{bmatrix}\right) \in T$ there can't be a different ordered pair in T having $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ as its 1st element.

To answer the next questions, the following fact is useful:

$$\begin{bmatrix} 9 & -31 & 98 & -84 & 1 & 0 & 0 \\ 2 & -5 & 18 & -13 & 0 & 1 & 0 \\ -4 & 14 & -44 & 38 & 0 & 0 & 1 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & 4 & 1 & -5 & 3 & -10 \\ 0 & 1 & -2 & 3 & -2 & 1 & -4 \\ 0 & 0 & 0 & 0 & 8 & -2 & 17 \end{bmatrix}$$

2D. What (if any) conditions are there on b_1, b_2 and b_3 so that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the image of T ?

$$8b_1 - 2b_2 + 17b_3 = 0$$

2E. Verify that $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ does in fact satisfy your conditions in (2D); show the work.

$$\begin{aligned} 8 \cdot 5 - 2 \cdot 3 + 17 \cdot (-2) &= \\ 40 - 6 - 34 &= \\ 40 - 40 &= 0 \text{ as expected.} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

where x_2 & x_4 are free.

2F. Find all \mathbf{x} such that $T(\mathbf{x}) = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$; express your answer in parametric ($\mathbf{x} = \mathbf{p} + v\mathbf{h}$) form.

this is equivalent to solving $A\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. The augment matrices above show if

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \text{ then } \begin{cases} x_1 = (-5 \cdot b_1 + 3b_2 - 10b_3) - 4x_3 - x_4 \\ x_2 = (-2b_1 + b_2 - 4b_3) + 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

becomes

$$\begin{cases} x_1 = (-5 \cdot 5 + 3 \cdot 3 - 10 \cdot (-2)) - 4x_3 - x_4 \\ x_2 = (-2 \cdot 5 + 3 - 4 \cdot (-2)) + 2x_3 - 3x_4 \\ x_3 = \text{free}, x_4 = \text{free} \end{cases}$$

$$\begin{cases} x_1 = 4 - 4x_3 - x_4 \\ x_2 = 1 + 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

3A. What does it mean to say that a set of vectors $S = \{c_1, c_2, \dots, c_p\}$ is linearly independent? (Give the definition).

We say a set $S = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_p\}$ is linearly independent

the only solution to $x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_p\vec{c}_p = \vec{0}$ is $x_1 = x_2 = \dots = x_p = 0$.

For the rest of this question let $S = \{c_1, c_2, c_3, c_4\}$ be the column vectors of $A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$.

3B. Find all solutions of $Ax = 0$. (Note the row reduced version of A appears in question (2)).

$$A \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -4x_3 - x_4 \\ x_2 = 2x_3 - 3x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

alternative notation $\vec{x} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$
 where x_3 & x_4 are free.

3C. Explicitly express $\vec{0}$ as a linear combination of c_1, c_2, c_3 and c_4 such that the weight of c_3 is one, or explain why this is impossible.

Since x_3 is free, we can choose it - the weight of \vec{c}_3 - to be 1.

Since x_4 is free, we can choose it to be 0 since we don't care if \vec{c}_4 appears in our L.C. Then $x_1 = -4 \cdot 1 - 0 = -4$ AND $x_2 = 2 \cdot 1 - 0 = 2$ so:

$$-4\vec{c}_1 + 2\vec{c}_2 + \vec{c}_3 = \vec{0}$$

(there are ∞ -many answers of course!)
 (by choosing x_4 differently)

3D. Express c_3 as a linear combination of the other columns, or explain why this cannot be done.

Solving (3C)'s answer for \vec{c}_3 gives

$$\vec{c}_3 = 4\vec{c}_1 - 2\vec{c}_2$$

3E. Is S linearly independent or linearly dependent? Explain your answer in terms of the definitions or any theorems we stated in class.

S is Linearly Dependent. ① At least one vector in S is a L.C. of the others, as shown in 3D.

② the matrix A of column vectors from S has more columns than rows

③ the set is not L.I. because 3C shows there are L.C.'s of $\vec{c}_1, \dots, \vec{c}_4$ adding to $\vec{0}$ yet not all weights are 0

— etc —

4. Define $T : X \rightarrow \mathbb{R}^3$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-ab} \\ 0 \\ 2a+3b \end{bmatrix}$, where X is the largest subset of \mathbb{R}^2 for which all the formulas can be computed.

4A. Find $T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-5} \\ 0 \\ 2 \cdot 1 + 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} \sqrt{4} \\ 0 \\ 2+15 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 17 \end{bmatrix}$

4B. Explicitly describe X .

$$X = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \mid 9-ab \geq 0 \right\} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \mid ab \leq 9 \right\}$$

4C. Find a smaller codomain for T than \mathbb{R}^3 ; explain your answer.

for any ab , $\sqrt{9-ab} \geq 0$; also $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ always has a "0" in the middle so a smaller

codomain could be $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x \geq 0, y=0, z \in \mathbb{R} \right\}$. There are other possible

4D. Give an explicit counter example that shows why T is not a linear transformation. Explain what you're doing.

many students used this:

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9} \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} \sqrt{9} \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$$

yet $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-1} \\ 0 \\ 2 \cdot 1 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} \sqrt{8} \\ 0 \\ 5 \end{bmatrix};$

Since $\begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} \neq \begin{bmatrix} \sqrt{8} \\ 0 \\ 5 \end{bmatrix}$ T is not a L.T. (others found examples where $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$ could be computed

4E. Bonus! Sketch the domain of T .

you need to plot " $b \leq 9/a$ " if $a > 0$

$b = \text{any } \#$ if $a = 0$

$b \geq 9/a$ if $a < 0$ [inequalities change sign when both sides are divided by a negative]

but $T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$ could not;

this was fine.

4F. Bonus! Sketch the image of T . (This will be tough).

the hard part is in showing that the codomain in 4C is actually the

image of T . This is not easy. It involves demonstrating that no matter how you pick $p \geq 0$ and r , there is a pair a, b for which $\sqrt{9-ab} = p$ AND $2a+3b = r$...

