1. Suppose the augmented matrix of the matrix equation $A\mathbf{x} = \mathbf{b}$ is: $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 2a \\ 0 & 1 & 3 & 0 & 4c \\ 0 & 0 & 0 & 3 & 6g \\ 0 & 0 & 0 & -5 & 8k \\ 0 & 0 & 0 & 0 & 10p \end{array}\right]$

1A. This augmented matrix is not quite in reduced row echelon form. Find the reduced row echelon form; *circle your final matrix.*

1B. Use the answer to (1A) to answer these questions: What conditions are there on a, c, g, k and p so that $A\mathbf{x} = \mathbf{b}$ has…

1B-i. …NO solutions. (tell me about all of a, c, g, k and p in *each* of these three parts).

1B-ii. …exactly one solution.

1B-iii. …infinitely many solutions.

1C: Suppose that a, c, g, k and p are in fact chosen so that $A\mathbf{x} = \mathbf{b}$ is consistent. Give the particular solution obtained by setting any and all free variables equal to 2; your answer will be in terms of a, c, g, k and p.

2. Suppose a linear transformation $T : \mathbf{R}^s \to \mathbf{R}^t$ is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is this matrix:

$$A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$$

2A. What are s and t? $s =$ ________ $t =$ ________

2B. Find $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

2C. Is $\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}\right) \in T$? *Explain your answer.*

To answer the next questions, the following fact is useful:

$$\begin{bmatrix} 9 & -31 & 98 & -84 & 1 & 0 & 0 \\ 2 & -5 & 18 & -13 & 0 & 1 & 0 \\ -4 & 14 & -44 & 38 & 0 & 0 & 1 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & 4 & 1 & -5 & 3 & -10 \\ 0 & 1 & -2 & 3 & -2 & 1 & -4 \\ 0 & 0 & 0 & 0 & 8 & -2 & 17 \end{bmatrix}$$

2D. What (if any) conditions are there on b_1, b_2 and b_3 so that $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the image of T?

2E. Verify that $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ does in fact satisfy your conditions in (2D); show the work.

2F. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$; express your answer in parametric ($\mathbf{x} = \mathbf{p} + \mathbf{v}_h$) form.

3A. What does it mean to say that a set of vectors $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\}$ is *linearly independent*? (Give the definition).

For the rest of this question let $S = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ be the column vectors of $A = \begin{bmatrix} 9 & -31 & 98 & -84 \\ 2 & -5 & 18 & -13 \\ -4 & 14 & -44 & 38 \end{bmatrix}$.

3B. Find all solutions of $A\mathbf{x} = \mathbf{0}$. (Note the row reduced version of A appears in question (2)).

3C. Explicitly express $\mathbf{0}$ as a linear combination of $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$ such that the weight of $\mathbf{c}_3$ is one, or explain why this is impossible.

3D. Express $\mathbf{c}_3$ as a linear combination of the other columns, or explain why this cannot be done.

3E. Is S linearly independent or linearly dependent? *Explain your answer* in terms of the definitions or any theorems we stated in class.

4. Define $T : X \to \mathbf{R}^3$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \sqrt{9-ab} \\ 0 \\ 2a+3b \end{bmatrix}$, where X is the largest subset of $\mathbf{R}^2$ for which all the formulas can be computed.

4A. Find $T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$.

4B. Explicitly describe X.

4C. Find a smaller codomain for T than $\mathbf{R}^3$; explain your answer.

4D. Give an explicit counter example that shows why T is not a linear transformation. Explain what you're doing.

4E. *Bonus!* Sketch the domain of T.

4F. *Bonus!* Sketch the image of T. (This will be tough).