

Math 105: Review for Exam I - Solutions

Graphical Relationships Between f , f' , and f''

f	positive	negative	increasing	decreasing	concave up	concave down
f'	X	X	positive	negative	increasing	decreasing
f''	X	X	X	X	positive	negative

1. Let $f(x) = \frac{x+1}{x^2-1}$. Note that $f(x) = \frac{x+1}{(x+1)(x-1)}$.

- (a) What is the natural domain of f ? $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$, i.e., all reals except $x = \pm 1$
- (b) Where is f discontinuous? $x = 1$ and $x = -1$
- (c) What is $\lim_{x \rightarrow 3} f(x)$? 0.5
- (d) What is $\lim_{x \rightarrow -1} f(x)$? -0.5
- (e) What is $\lim_{x \rightarrow 1} f(x)$? undefined
- (f) What is $\lim_{x \rightarrow 1^+} f(x)$? ∞

2. Let $f(x) = 3x^2 - 2x$.

- (a) Compute the average rate of change of f on the interval $[2, 2.1]$.

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.03 - 8}{0.1} = 10.3$$

- (b) Using the limit definition of the derivative, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h - 2)}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h - 2) \\ &= 6x - 2 \end{aligned}$$

- (c) Find the equation of the tangent line to f at $x = 2$.

We want $y = mx + b$. $m = f'(2) = 6 \cdot 2 - 2 = 10$, so $y = 10x + b$.

When $x = 2$, $y = f(2) = 3 \cdot 2^2 - 2 \cdot 2 = 8$.

Thus, $8 = 10 \cdot 2 + b$, so $b = -12$ and we have $y = 10x - 12$.

- (d) How would the derivative of $g(x) = f(x) + 5$ compare to $f'(x)$?

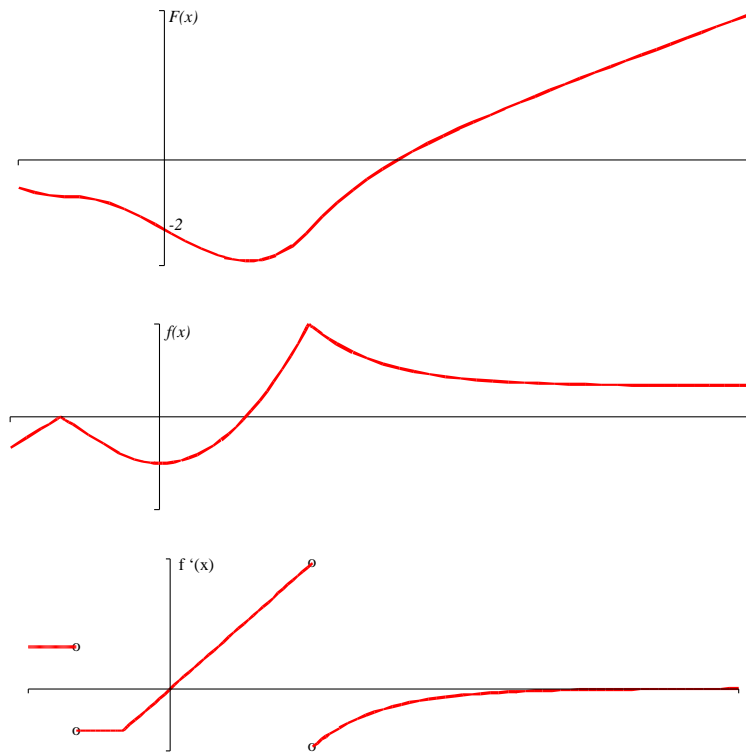
The graph of $y = f(x) + 5$ is the graph of $y = f(x)$ shifted vertically by 5 units, but this has no effect on the slope of the graph, so $g'(x) = f'(x)$.

- (e) How would the derivative of $h(x) = 5f(x)$ compare to $f'(x)$?

The graph of $y = 5f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of 5; this also results in slopes that are 5 times greater at any given x -value. Thus, $h'(x) = 5f'(x)$.

Note that we get the same result by considering our derivative rule $\frac{d}{dx} [kf(x)] = kf'(x)$ where $k = 5$.

3. Given the graph of f , sketch a graph of f' and a graph of F , an antiderivative of f such that $F(0) = -2$.



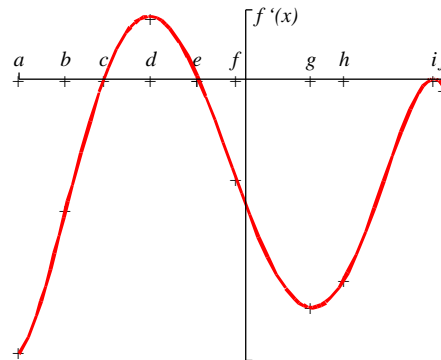
Note: The concave up portion in the middle of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f , your graph of f' may be concave up/down there.

4. Shown below is a graph of f' on its entire domain. The graph is NOT f .

At which x -value(s)

- (a) does f have a stationary point? c, e, i
- (b) does f have a local max? e
- (c) does f have a local min? c
- (d) does f' have a stationary point? d, g, i
- (e) does f' have a local max? d, i
- (f) does f' have a local min? g
- (g) is f greatest? a
- (h) is f least? j
- (i) is f' greatest? d
- (j) is f' least? a
- (k) is f'' greatest? b
- (l) is f'' least? f (or e - it's close)

- (b) f decreasing? $[a, c) \cup (e, j]$
- (c) f' increasing? $[a, d) \cup (g, i)$
- (d) f' decreasing? $(d, g) \cup (i, j]$
- (e) f concave up? $[a, d) \cup (g, i)$
- (f) f concave down? $(d, g) \cup (i, j]$



On what interval(s) is

- (a) f increasing? (c, e)

5. Suppose that $T(t)$ gives the temperature in Lewiston as a function of time. In each of the following situations, determine if the signs of T , T' , and T'' are positive, negative, zero, or unknown.

(a) **The temperature is 60 degrees and falling steadily.**

The temperature is 60, so we know T is positive.

The temperature is *falling*, so we know T' is negative.

The temperature is falling *steadily*, so we know the graph is linear, and T'' is zero.

(b) **The temperature is rising more and more slowly.**

We don't know whether the temperature is above or below zero, so the sign of T is unknown.

The temperature is *rising*, so we know T' is positive.

The temperature is rising *more and more slowly*, so we know the graph of T is concave down, and T'' is negative.

(c) **The temperature is -5 degrees and rising.**

The temperature is -5, so we know T is negative.

The temperature is *rising*, so we know T' is positive.

We don't know the concavity of the graph of T , so the sign of T'' is unknown.

6. An object has vertical velocity $v(t) = t^2 - 5t + 4$ feet per second on the interval $[0, 5]$. A positive velocity indicates the object is ascending, and a negative velocity indicates it is descending. At time $t = 0$, the object is 50 feet above ground.

(a) **When is the object ascending? When is it descending?**

$v(t) = (t - 1)(t - 4) = 0$ when $t = 1$ and $t = 4$.

On the intervals $[0, 1)$ and $(4, 5]$, v' is positive, so the object is ascending during these times.

On the interval $(1, 4)$, v' is negative, so the object is descending during this time.

(b) **When is the object's acceleration most positive?**

The acceleration is $v'(t) = 2t - 5$. This is an increasing function, so it is most positive at the right endpoint of the interval, or at $t = 5$.

(c) **What is the greatest height the object reaches?**

From part (a), we know that the object is rising until $t = 1$ and again until $t = 5$, so the greatest height must occur at one of those times.

We can find the height by taking the antiderivative of $v(t)$, which gives $h(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 4t + C$.

Since we know $h(0) = 50$, we see that $C = 50$.

Now we compare the heights at $t = 1$ and $t = 5$.

$h(1) = 311/6$ and $h(5) = 295/6$, so the maximum height is $311/6 = 51.8\bar{3}$ feet.

7. Find the derivatives of the following.

(a) $y = 2 + 3x + x^4 + 5x^6$

$$y' = 3 + 4x^3 + 30x^5$$

(b) $y = \sqrt[6]{x} + \frac{1}{x^6} + \frac{x}{6} + \frac{6}{x} + \frac{\pi}{6} + 6^{1/2} + \sqrt{6x^{1/6}}$

First, rewrite y to make it easier to apply our derivative rules:

$$y = x^{1/6} + x^{-6} + \frac{1}{6} \cdot x + 6x^{-1} + \frac{\pi}{6} + 6^{1/2} + 6^{1/2}x^{1/12}$$

$$\text{Note that above } \sqrt{6x^{1/6}} = \sqrt{6}\sqrt{x^{1/6}} = 6^{1/2}(x^{1/6})^{1/2} = 6^{1/2}x^{1/12}.$$

$$y' = \frac{1}{6}x^{-5/6} + (-6)x^{-7} + \frac{1}{6} + (6)(-1)(x^{-2}) + 0 + 0 + 6^{1/2} \frac{1}{12}x^{-11/12}$$

To clean this up, we use exponent rules: $x^{-a} = \frac{1}{x^a}$ and $x^{a/b} = \sqrt[b]{x^a}$

$$y' = \frac{1}{6\sqrt[6]{x^5}} - \frac{6}{x^7} + \frac{1}{6} - \frac{6}{x^2} + \frac{\sqrt{6}}{12\sqrt[12]{x^{11}}}$$

8. Find antiderivatives of the following.

(a) $y = \pi + 3x^2$

$$\text{antiderivative} = \pi x + x^3 + C$$

(b) $y = 4x^5 - \frac{1}{x^6} = 4x^5 - x^{-6}$

$$\text{antiderivative} = \frac{4x^6}{6} - \frac{x^{-5}}{-5} + C = \frac{2x^6}{3} + \frac{1}{5x^5} + C$$

9. Is $y = 5x^3$ a solution to the differential equation $xy' - 3y = 0$?

The question asks whether, when we plug in y and y' , $xy' - 3y$ will equal 0.

We are given $y = 5x^3$, so $y' = 15x^2$.

$$xy' - 3y \stackrel{?}{=} 0$$

$$x \cdot 15x^2 - 3 \cdot 5x^3 \stackrel{?}{=} 0 \quad \text{Substitute } y \text{ and } y' \text{ in the appropriate places.}$$

$$15x^3 - 15x^3 \stackrel{?}{=} 0$$

$$0 = 0 \quad \text{This is true.}$$

So, $y = 5x^3$ is a solution to the given differential equation.

10. Solve the IVP (initial value problem) $f'(x) = -1 + x^3$ if $f(2) = 13$.

We begin by finding the antiderivative of $f'(x)$: $f(x) = -x + \frac{x^4}{4} + C$.

Now we plug in the 2 and the 13 to find the value of C .

$$13 = -2 + \frac{2^4}{4} + C$$

$$13 = -2 + 4 + C$$

$$11 = C$$

So, the solution to this IVP is $f(x) = -x + \frac{x^4}{4} + 11$.