Mid-term Exam #1
MATH 205, Fall 2014

Name: ____

Instructions: Please answer as many of the following questions as possible. Show all of your work and give complete explanations when requested. Write your final answer clearly. No calculators or cell phones are allowed.

This exam has 5 problems and 100 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	**100**	

1. (20 points) Consider the following linear system of equations:

$$\begin{cases} x_1 & & + & 2x_3 & + & 4x_4 & = & -8 \\ & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + \; 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - \; x_2 & + & 3x_3 & + & 4x_4 & = & -6 \end{cases}$$

(a) (10 points) Find the reduced echelon form of the augmented matrix associated with this system.

(b) (5 points) Write the general solution to the system in parametric vector form.

(c) (5 points) Write the general solution to the corresponding homogeneous system of equations in parametric vector form.

Solution:

(a)

$$\begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -6 \end{bmatrix} \quad r_3 \mapsto -3r_1 + r_3 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -6 \end{bmatrix}$$

$$r_3 \mapsto -4r_2 + r_3 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 4 & -6 \end{bmatrix}$$

$$r_4 \mapsto r_2 + r_4 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

$$r_3 \leftrightarrow r_2 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r_3 \mapsto (1/3)r_3 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r_2 \mapsto r_3 + r_2 \quad \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r_1 \mapsto -4r_3 + r_1 \quad \begin{bmatrix} 1 & 0 & 2 & 0 & -8 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad t \text{ in } \mathbb{R}$

(c) $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad t \text{ in } \mathbb{R}$

2. (20 points) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

   (a) (10 points) Is the set $\{\mathbf{u}, \mathbf{v}\}$ linearly independent or dependent? Justify your answer.

   (b) (10 points) Find all values of h such that $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

SOLUTION:

(a) The set $\mathbf{u}$, $\mathbf{v}$ is linearly independent because by inspection I see that $\mathbf{u}$ is not a multiple of $\mathbf{v}$.

(b) Find h so that there exist c_1, c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{w}$. Write this vector equation as an augmented matrix and row reduce to find h so that the system is consistent:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & h \end{bmatrix} \quad r_2 \mapsto -2r_1 + r_2 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & h \end{bmatrix}$$
$$r_3 \leftrightarrow r_2 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & h \\ 0 & 2 & -1 \end{bmatrix}$$
$$r_3 \mapsto -2r_2 + r_3 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & h \\ 0 & 0 & -1-2h \end{bmatrix}.$$

From the last matrix above we see that the vector equation will have a solution when $h = -1/2$. Then $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ when $h = -1/2$.

3. (20 points) Let $T(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{bmatrix}.$$

(a) (10 points) Show, using the definition of linear transformation, that T is a linear transformation.

(b) (5 points) Find A such that $T(\mathbf{x}) = A\mathbf{x}$. Show that $A^2 = A$.

(c) (5 points) Is T a mapping from $\mathbb{R}^2$ onto $\mathbb{R}^2$? *Give a one sentence explanation supporting your answer.* If T is not onto, find a vector $\mathbf{b}$ that is *not* in the range of T.

SOLUTION:

(a) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and c be a real number. Check two properties of T.

i. Show that for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \frac{u_1+v_1+u_2+v_2}{2} \\ \frac{u_1+v_1+u_2+v_2}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{u_1+u_2}{2} + \frac{v_1+v_2}{2} \\ \frac{u_1+u_2}{2} + \frac{v_1+v_2}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{u_1+u_2}{2} \\ \frac{u_1+u_2}{2} \end{bmatrix} + \begin{bmatrix} \frac{v_1+v_2}{2} \\ \frac{v_1+v_2}{2} \end{bmatrix} \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

ii. Show that for all $\mathbf{u}$ in $\mathbb{R}^2$ and all c in $\mathbb{R}$, $T(c\mathbf{u}) = cT(\mathbf{u})$:

$$\begin{aligned} T(c\mathbf{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \frac{cu_1+cu_2}{2} \\ \frac{cu_1+cu_2}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{c(u_1+u_2)}{2} \\ \frac{c(u_1+u_2)}{2} \end{bmatrix} \\ &= c\begin{bmatrix} \frac{u_1+u_2}{2} \\ \frac{u_1+u_2}{2} \end{bmatrix} \\ &= cT(\mathbf{u}) \end{aligned}$$

(b) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1/4 + 1/4 & 1/4 + 1/4 \\ 1/4 + 1/4 & 1/4 + 1/4 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A \end{aligned}$$

(c) T is not an onto mapping. The reduced echelon form of the matrix A is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. This matrix does not have a pivot in every row so therefore T is not a mapping from $\mathbb{R}^2$ onto $\mathbb{R}^2$. *Alternative answer:* The columns of A do not span $\mathbb{R}^2$, therefore T is not a mapping from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

The vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in the range of T (there are many correct answers). If I augment the matrix A with the vector $\mathbf{b}$ and row reduce, the reduced echelon form of the augmented matrix is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ which shows that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. This, in turn means that there is no vector $\mathbf{x}$ in $\mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

4. (20 points)

(a) (10 points) Consider the following system of equations:

$$\begin{aligned} x_1 + hx_2 &= 1 \\ 2x_1 + 3x_2 &= k. \end{aligned}$$

Choose h and k so that the system has

i. no solution;

ii. a unique solution;

iii. many solutions.

(b) (10 points) Let A be an $n \times n$ matrix. Explain why the columns of A are linearly dependent when $\det A = 0$. *Write your answer in complete sentences.*

SOLUTION:

(a) Build the augment matrix associated the the system and perform the row reduction algorithm to obtain echelon form:

$$\begin{bmatrix} 1 & h & 1 \\ 2 & 3 & k \end{bmatrix} \quad r_2 \mapsto -2r_1 + r_2 \quad \begin{bmatrix} 1 & h & 1 \\ 0 & 3-2h & k-2 \end{bmatrix} \tag{1}$$

i. From (1), the system will have no solution if $3-2h = 0$ and $k-2 \neq 0$. Or, if $h = 3/2$ and $k \neq 2$. This will make the system represented by (1) inconsistent.

ii. From (1), the system will have a unique solution if $3 - 2h \neq 0$ and $k - 2$ is any real number. Or, if $h \neq 3/2$ and k is any real number.

iii. From (1), the system will have many solutions is if there is a free variable, meaning that $3 - 2h = 0$ and $k - 2 = 0$. Or, equivalently, if $h = 3/2$ and $k = 2$.

(b) Suppose that $\det A = 0$. Then A is not invertible. By the Invertible Matrix Theorem, this means that the homogenous equation $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions. This, in turn, means that the columns of A are linearly dependent.

5. (20 points)

(a) (10 points) Compute $\det\left(B^6\right)$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

(b) (10 points) Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

SOLUTION:

(a) By the multiplicative property of determinants, we know that $\det\left(B^6\right) = \left(\det B\right)^6$. So we calculate $\det B$ using the method of cofactors, expanding across the first row:

$$\begin{aligned} \det B &= 1 \cdot \left(\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right) - 0 \cdot \left(\det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) + 1 \cdot \left(\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) \\ &= 1\big(1-4\big) + 1\big(2-1\big) \\ &= -3 + 1 \\ &= -2. \end{aligned}$$

Then $\left(\det B\right)^6 = (-2)^6 = 64$.

(b) By the multiplicative property of determinants,

$$\det\left(AA^{-1}\right) = \det A \cdot \det A^{-1}.$$

I also know that $AA^{-1} = I_n$, the $n \times n$ identity matrix, and $\det I_n = 1$ (this can be seen because I_n is a diagonal matrix). So

$$\det A \cdot \det A^{-1} = 1.$$

Solving for $\det A$, we get $\det A^{-1} = \frac{1}{\det A}$, where is it okay to divide by $\det A$ because we know that $\det A \neq 0$ by the fact that A is invertible.