MATH 205_A FALL 2008

EXAM I 10/3/2008

NAME (PRINT)

Suggested soln's

DO NOT WRITE HERE

1	
2	
3	
4	
5	
total	

① show all your work in the SPACES PROVIDED

② indicate your final answer.

③ be NEAT. write NEATLY

GOOD LUCK!

Here [redacted] are facts you may find useful:

For problem 1:

The RREF of
$$\left[\begin{array}{rrrr|rrrrr} 3 & -11 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 4 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & -3 & -4 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & -7 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$
is
$$\left[\begin{array}{rrrr|rrrrr} 1 & 0 & 0 & 0 & 0 & 3 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -18 & -3 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -16 & -4 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 16 & 5 & 3 & 0 \end{array}\right]$$

For problems 2 and 3:

$$\left[\begin{array}{rrrrrrrrr} -8 & -2 & -1 & -6 & 2 & 1 & 0 & 0 & 0 \\ 20 & 5 & 4 & 12 & -4 & 0 & 1 & 0 & 0 \\ -9 & -2 & -3 & -3 & 1 & 0 & 0 & 1 & 0 \\ 24 & 6 & 9 & 6 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$
is row equivalent to
$$\left[\begin{array}{rrrrrrrrr} 1 & 0 & 0 & 1 & -1/3 & 0 & 0 & -1 & -1/3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3/7 & 4 & 8/7 \\ 0 & 0 & 1 & -2 & 2/3 & 0 & -2/7 & 0 & 5/21 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4/7 & 0 & -1/7 \end{array}\right]$$

1. Let $B = \begin{bmatrix} 3 & -11 & 1 & -3 \\ 1 & 0 & 1 & -1 \\ -2 & 4 & -1 & 2 \\ -3 & -3 & -4 & 3 \\ 1 & -7 & 0 & -2 \end{bmatrix}$, let $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ and $\mathbf{c}_4$ be the column vectors in B, and let $T : \mathbf{R}^j \to \mathbf{R}^q$ be the linear transformation which has B as its standard matrix.

1A. What are the values of j and q? $j = \boxed{4}$ $q = \boxed{5}$

1B. Do the columns $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ and $\mathbf{c}_4$ of B form a linearly independent set? Explain your answer in terms of the definition of linear independence.

YES. The RREF of B shows there are no free variables, hence, the only solution of $B\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. This means the only way to form a linear combination $x_1\vec{c}_1 + \cdots + x_4\vec{c}_4$ of the columns of B which yields $\vec{0}$ is by choosing $x_1 = x_2 = x_3 = x_4 = 0$

1C. Does $B\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b}$ in $\mathbf{R}^q$? Explain your answer.

NO. The RREF of $[B \mid \vec{b}]$ has last row:

$0 = b_1 + 16b_2 + 5b_3 + 3b_4 + 0b_5$; thus $B\vec{x} = \vec{b}$ has a soln $\Leftrightarrow$ this condition is satisfied by $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_5 \end{bmatrix}$ $\leftarrow$ (for example, $B\vec{x} = \begin{bmatrix} \vdots \end{bmatrix}$ has no solution)

1D. Find all solutions $\mathbf{v}_h$ of the homogeneous system $B\mathbf{x} = \mathbf{0}$.

Since the system has only the trivial solution, $\vec{v}_h$ is $\vec{0}$.

1E. Calculate $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$ and call the result "a". (So this is an instruction to add columns 1 and 3.) $\mathbf{a} = \boxed{\begin{bmatrix} 4 \\ 2 \\ -3 \\ -7 \\ 1 \end{bmatrix}}$

We get $\vec{a}$ as "in the box" $\to$

1F. Find *all* solutions of $B\mathbf{x} = \mathbf{a}$ (see part 1E).

Since $B\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \vec{a}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution. Since $\vec{v}_h = \vec{0}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is in fact the ONLY solution.

1G. Is T *onto* $\mathbf{R}^q$? Explain.

No ; $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{bmatrix}$ is in the image of $T \Leftrightarrow 0 = b_1 + 16b_2 + 5b_3 + 3b_4$, so there are some vectors $\vec{b} \in \mathbb{R}^5$ which are not in the image of T; hence T is not onto.

2. Let $D = \begin{bmatrix} -8 & -2 & -1 & -6 & 2 \\ 20 & 5 & 4 & 12 & -4 \\ -9 & -2 & -3 & -3 & 1 \\ 24 & 6 & 9 & 6 & -2 \end{bmatrix}$

2A. Find all conditions on b_1, b_2, b_3 and b_4 which must be satisfied in order for $\mathbf{b} = (b_1,\ b_2,\ b_3,\ b_4)$ (written sideways to save space) to be in the span of the columns of D.

The supplied row reduction of $[D|\vec{b}]$ to RREF tells us $\vec{b}$ is in the span of the columns of $D \Leftrightarrow 0 = b_1 + \frac{4}{7}b_2 + 0b_3 - \frac{1}{7}b_4$, or $\boxed{0 = 7b_1 + 4b_2 + 0b_3 - b_4}$ ⊗

2B. Find all values of b_2 for which $\mathbf{b} = (17,\ b_2,\ 5,\ -5)$ (written sideways...) has a solution in the matrix equation $D\mathbf{x} = \mathbf{b}$.

From ⊗ we have

$$0 = 7\cdot 17 + 4b_2 - (-5)$$
$$0 = (119+5) + 4b_2$$
$$-b_2 = 124/4 = 31 \quad \therefore \quad \boxed{b_2 = -31}$$

2C. Find all values of b_3 for which $\mathbf{c} = (17,\ -31,\ b_3,\ -5)$ (sideways again) has a solution in $D\mathbf{x} = \mathbf{c}$.

It doesn't matter what b_3 is, since $0 = 7\cdot 17 + 4(-31) + 0b_3 - (-5)$ is true for any b_3. (But you <u>do</u> have to point out that $0 = 7\cdot 17 - 4\cdot 31 - -5$!)

2D. Verify that $\mathbf{d} = (7,\ -7,\ 8,\ 21)$ (sideways again) satisfies any/all conditions found in 2A. (Show your calculations)

Let's check: $0 \stackrel{?}{=} (7\cdot 7) + (4\cdot -7) + (0\cdot 8) - 21$

$$= 49 - 28 - 21$$
$$= 49 - 49 = 0 \checkmark$$

2E. Write all solutions of $D\mathbf{x} = \mathbf{d}$ in the form $\mathbf{p} + \mathbf{v}_h$ where p is a particular solution of $D\mathbf{x} = \mathbf{d}$ and $\mathbf{v}_h$ is all solutions of the associated homogeneous equation.

The supplied RREF tells us that

$$\begin{cases} x_1 = (-1\cdot b_3 - \frac{1}{3}b_4) - x_4 + \frac{1}{3}x_5 \\ x_2 = (\frac{3}{7}b_2 + 4b_3 + \frac{8}{7}b_4) \\ x_3 = (-\frac{2}{7}b_2 + \frac{5}{21}b_4) + 2x_4 - \frac{2}{3}x_5 \\ \text{where } x_4 \ \&\ x_5 \text{ are free and } \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 8 \\ 21 \end{bmatrix} \end{cases}$$

so $\begin{cases} x_1 = (-8-7) - x_4 + \frac{1}{3}x_5 \\ x_2 = (-3+32+24) \\ x_3 = (2+5) + 2x_4 - \frac{2}{3}x_5 \end{cases}$

thus: $\vec{x} = \begin{bmatrix} -15 \\ 53 \\ 7 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1/3 \\ 0 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}$

2F. Find two more particular (or, explicit) solutions of $D\mathbf{x} = \mathbf{d}$. Show how you got them.

Any choices of x_4 and x_5 in this expression will yield more soln's. For example, $x_4 = 1$, $x_5 = 0$ gives the solution $\begin{bmatrix} -16 \\ 53 \\ 9 \\ 1 \\ 0 \end{bmatrix}$

3. Again let $D = \begin{bmatrix} -8 & -2 & -1 & -6 & 2 \\ 20 & 5 & 4 & 12 & -4 \\ -9 & -2 & -3 & -3 & 1 \\ 24 & 6 & 9 & 6 & -2 \end{bmatrix}$. Let the column vectors of D be labeled $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_5$

3A. Express $\mathbf{d}_2$ as a linear combination of the other columns of D or explain why this is impossible.

The solutions of $D\vec{x} = \vec{0}$ are $\vec{x} = x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1/3 \\ 0 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}$ where x_4 and x_5 are free (see previous page)

(✱) → In particular, no solution exists in which the weight in front of $\vec{d}_2$ is nonzero.
But suppose $\vec{d}_2$ is a L.C. of the other columns; then $\vec{d}_2 = \alpha_1\vec{d}_1 + \alpha_3\vec{d}_3 + \alpha_4\vec{d}_4 + \alpha_5\vec{d}_5$,
or, $\vec{0} = \alpha_1\vec{d}_1 + (-1)\vec{d}_2 + \alpha_3\vec{d}_3 + \alpha_4\vec{d}_4 + \alpha_5\vec{d}_5$, which contradicts (✱),
since -1 is a non-zero weight. $\therefore$ $\vec{d}_2$ is NOT a L.C. of the other columns.

3B. Express $\mathbf{d}_3$ as a linear combination of the other columns of D or explain why this is impossible.

We begin with a sol'n of $D\vec{x} = \vec{0}$ that does have a nonzero weight in front of $\vec{d}_3$,
we can take $x_4 = 1$ and $x_5 = 0$ in $\vec{v}_h$ to get $x_1 = -1$, $x_2 = 0$, $x_3 = 2$,
so $\vec{0} = -\vec{d}_1 + 0\vec{d}_2 + 2\vec{d}_3 + 1\vec{d}_4 + 0\vec{d}_5$; "solving" for $\vec{d}_3$ we get
$\vec{d}_3 = \frac{1}{2}\vec{d}_1 - \frac{1}{2}\vec{d}_4$. (other combinations are possible through different choices of x_4 and x_5)

3C. Write the vector $10\mathbf{d}_1 + 20\mathbf{d}_2 + 30\mathbf{d}_3 + 40\mathbf{d}_4 + 50\mathbf{d}_5$ as a linear combination of $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_4$ and $\mathbf{d}_5$ (that is, without using $\mathbf{d}_3$) or explain why this is impossible.

$$10\vec{d}_1 + 20\vec{d}_2 + 30\vec{d}_3 + 40\vec{d}_4 + 50\vec{d}_5$$
$$= 10\vec{d}_1 + 20\vec{d}_2 + 30\left(\tfrac{1}{2}\vec{d}_1 - \tfrac{1}{2}\vec{d}_4\right) + 40\vec{d}_4 + 50\vec{d}_5$$
$$= 25\vec{d}_1 + 20\vec{d}_2 + 25\vec{d}_4 + 50\vec{d}_5$$

other L.C.'s are possible if x_4 & x_5 are chosen differently in part 3B. For example, choosing $x_4 = x_5 = 1$ gives $25\vec{d}_1 + 20\vec{d}_2 + 17.5\vec{d}_4 + 27.5\vec{d}_5$

4A. Suppose that $T : \mathbf{R}^a \to \mathbf{R}^b$ is a transformation. Define what it means for T to be a *linear* transformation. Your answer will include phrases like "for all $\mathbf{v}$ in ...", etc.

Two conditions must be met:

① $T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^a$

② $T(\alpha\vec{u}) = \alpha\, T(\vec{u})$ for all vectors $\vec{u} \in \mathbb{R}^a$ and scalars $\alpha \in \mathbb{R}$.

4B. Suppose $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \left(\begin{bmatrix} x - 2y + 4z \\ 1 \\ 0 \end{bmatrix}\right)$. Following the approved way we did so in class, show why T is *not* a linear transformation. Does either part of the L.T. definition hold?

The approved way to show "NOT" something is to provide a specific counter example. We offer:

Does $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + T\left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$?

LHS is $\begin{bmatrix} 1-4+12 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4-10+24 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ 2 \\ 0 \end{bmatrix}$

RHS is $T\left(\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 5-14+36 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ 1 \\ 0 \end{bmatrix}$ Close, yes, but not equal!

in fact, neither part holds: for example

$T\left(3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3-6+12 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$

whereas $3\,T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 1-2+4 \\ 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix}$ again, close but unequal