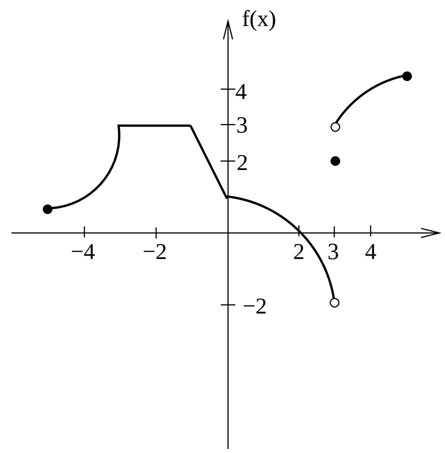
Math 105: Review for Exam I - Solutions

1. **Let $f(x) = 3 + \sqrt{x+5}$.**
   (a) **What is the natural domain of f?** $[-5, \infty)$, which means all reals greater than or equal to 5
   (b) **What is the range of f?** $[3, \infty)$, which means all reals greater than or equal to 3

2. **For the graph of f shown, answer the following.**
   (a) **Evaluate the following.**
      i. $f'(-2) = 0$
      ii. $f(3) = 2$
      iii. $\lim_{x \to 3^-} f(x) = -2$
      iv. $\lim_{x \to 3^+} f(x) = 3$
      v. $\lim_{x \to 3} f(x)$ does not exist
      vi. $\lim_{x \to 2} f(x) = 0$



   (b) **Where is f discontinuous?** at $x = 3$
   (c) **Where does f' fail to exist?**
   at $x = -3, -1, 0, 3$

3. **Let $f(x) = 3x^2 - 2x$.**
   (a) **Compute the average rate of change of f on the interval [2,2.1].**

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.03 - 8}{0.1} = 10.3$$

   (b) **Using the limit definition of the derivative, find $f'(x)$.**

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{provided this limit exists} \\ &= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h} \\ &= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\ &= \lim_{h \to 0} \frac{h(6x + 3h - 2)}{h} \\ &= \lim_{h \to 0} (6x + 3h - 2) \\ &= 6x - 2 \end{aligned}$$

   (c) **Find the equation of the tangent line to f at $x = 2$.**
   We want $y = mx + b$. $m = f'(2) = 6 \cdot 2 - 2 = 10$, so $y = 10x + b$.
   When $x = 2$, $y = f(2) = 3 \cdot 2^2 - 2 \cdot 2 = 8$.
   Thus, $8 = 10 \cdot 2 + b$, so $b = -12$ and we have $y = 10x - 12$.

   (d) **How would the derivative of $g(x) = f(x) + 5$ compare to $f'(x)$?**
   The graph of $y = f(x) + 5$ is the graph of $y = f(x)$ shifted vertically by 5 units, but this has no effect on the slope of the graph, so $g'(x) = f'(x)$.

(e) **How would the derivative of $h(x) = 5f(x)$ compare to $f'(x)$?**

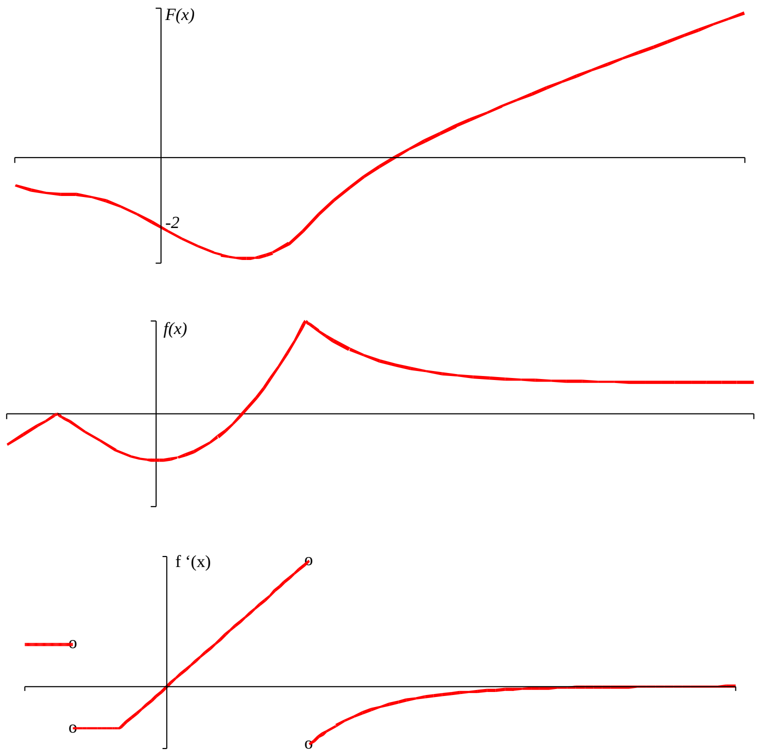
The graph of $y = 5f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of 5; this also results in slopes that are 5 times greater at any given x-value. Thus, $h'(x) = 5f'(x)$.

Note that we get the same result by considering our derviative rule $\frac{d}{dx}[kf(x)] = kf'(x)$ where $k = 5$.

4. **Fill in the table showing the graphical relationships between f, f', and f''.**

f	positive	negative	increasing	decreasing	concave up	concave down
f'	X	X	positive	negative	increasing	decreasing
f''	X	X	X	X	positive	negative

5. **Given the graph of f, sketch a graph of f' and a graph of F, an antiderivative of f such that $F(0) = -2$.**



Note: The concave up portion in the middle of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f, your graph of f' may be concave up/down there.

6. **Shown below is a graph of f' on its entire domain. The graph is NOT f.**

At which x-value(s)

(a) **does f have a stationary point?** c, e, j

(b) **does f have a local max?** e

(c) **does f have a local min?** c

(d) **does f' have a stationary point?** d, h, j

(e) **does f' have a local max?** d, j

(f) **does f' have a local min?** h

(g) **is f greatest?** a

(h) **is f least?** k

(i) **is f' greatest?** d

(j) **is f' least?** a

(k) **is f'' greatest?** b

(l) **is f'' least?** f

On what interval(s) is

(a) **f increasing?** (c, e)

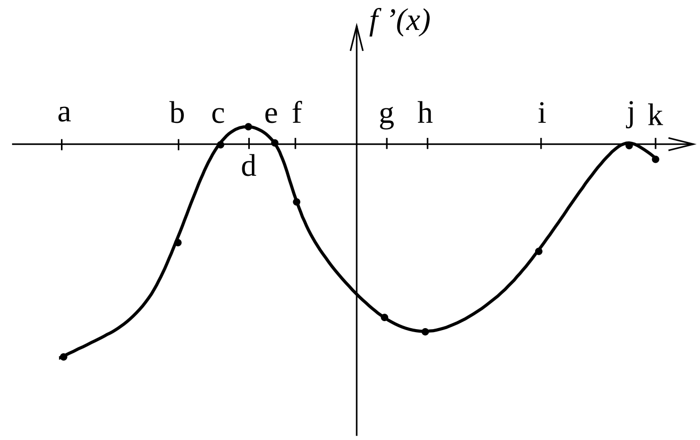
(b) **f decreasing?** $[a, c) \cup (e, k]$

(c) **f' increasing?** $[a, d) \cup (h, j)$

(d) **f' decreasing?** $(d, h) \cup (j, k]$

(e) **f concave up?** $[a, d) \cup (h, j)$

(f) **f concave down?** $(d, h) \cup (j, k]$



7. **Suppose that $T(t)$ gives the temperature in Lewiston as a function of time. In each of the following situations, determine if the signs of T, T', and T'' are positive, negative, zero, or unknown.**

(a) **The temperature is 60 degrees and falling steadily.**
The temperature is 60, so we know T is positive.
The temperature is *falling*, so we know T' is negative.
The temperature is falling *steadily*, so we know the graph is linear, and T'' is zero.

(b) **The temperature is rising more and more slowly.**
We don't know whether the temperature is above or below zero, so the sign of T is unknown.
The temperature is *rising*, so we know T' is positive.
The temperature is rising *more and more slowly*, so we know the graph of T is concave down, and T'' is negative.

(c) **The temperature is -5 degrees and rising.**
The temperature is -5, so we know T is negative.
The temperature is *rising*, so we know T' is positive.
We don't know the concavity of the graph of T, so the sign of T" is unknown.

8. **An object has vertical velocity $v(t) = t^2 - 5t + 4$ feet per second on the interval $[0, 5]$. A positive velocity indicates the object is ascending, and a negative velocity indicates it is descending. At time $t = 0$, the object is 50 feet above ground.**

(a) **When is the object ascending? When is it descending?**
$v(t) = (t - 1)(t - 4) = 0$ when $t = 1$ and $t = 4$.
On the intervals $[0, 1)$ and $(4, 5]$, v' is positive, so the object is ascending during these times.

On the interval $(1, 4)$, v' is negative, so the object is descending during this time.

(b) **When is the object's acceleration most positive?**
The acceleration is $v'(t) = 2t - 5$. This is an increasing function, so it is most positive at the right endpoint of the interval, or at $t = 5$.

(c) **What is the greatest height the object reaches?**
From part (a), we know that the object is rising until $t = 1$ and again until $t = 5$, so the greatest height must occur at one of those times.

We can find the height by taking the antiderivative of $v(t)$, which gives $h(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 4t + C$.
Since we know $h(0) = 50$, we see that $C = 50$.
Now we compare the heights at $t = 1$ and $t = 5$.
$h(1) = 311/6$ and $h(5) = 295/6$, so the maximum height is $311/6 = 51.8\overline{3}$ feet.

9. **Find the derivatives of the following.**

(a) $\boldsymbol{y = 2 + 3x + x^4 + 5x^6}$
$y' = 3 + 4x^3 + 30x^5$

(b) $\boldsymbol{y = \sqrt[6]{x} + \frac{1}{x^6} + \frac{x}{6} + \frac{6}{x} + \frac{\pi}{6} + 6^{1/2} + \sqrt{6x^{1/6}}}$
First, rewrite y to make it easier to apply our derivative rules:
$y = x^{1/6} + x^{-6} + \frac{1}{6} \cdot x + 6x^{-1} + \frac{\pi}{6} + 6^{1/2} + 6^{1/2}x^{1/12}$

Note that above $\sqrt{6x^{1/6}} = \sqrt{6}\sqrt{x^{1/6}} = 6^{1/2}(x^{1/6})^{1/2} = 6^{1/2}x^{1/12}$.

$$y' = \frac{1}{6}x^{-5/6} + (-6)x^{-7} + \frac{1}{6} + (6)(-1)(x^{-2}) + 0 + 0 + 6^{1/2}\frac{1}{12}x^{-11/12}$$

To clean this up, we use exponent rules: $x^{-a} = \frac{1}{x^a}$ and $x^{a/b} = \sqrt[b]{a}$

$$y' = \frac{1}{6\sqrt[6]{x^5}} - \frac{6}{x^7} + \frac{1}{6} - \frac{6}{x^2} + \frac{\sqrt{6}}{12\sqrt[12]{x^{11}}}$$

10. **Find antiderivatives of the following.**

(a) $\boldsymbol{y = \pi + 3x^2}$
antiderivative $= \pi x + x^3 + C$

(b) $\boldsymbol{y = 4x^5 - \frac{1}{x^6}} = 4x^5 - x^{-6}$
antiderivative $= \frac{4x^6}{6} - \frac{x^{-5}}{-5} + C = \frac{2x^6}{3} + \frac{1}{5x^5} + C$

11. **Is $\boldsymbol{y = 5x^3}$ a solution to the differential equation $\boldsymbol{xy' - 3y = 0}$?**

The question asks whether, when we plug in y and y', $xy' - 3y$ will equal 0.
We are given $y = 5x^3$, so $y' = 15x^2$.

$$
\begin{aligned}
xy' - 3y &\stackrel{?}{=} 0 \\
x \cdot 15x^2 - 3 \cdot 5x^3 &\stackrel{?}{=} 0 && \text{Substitute } y \text{ and } y' \text{ in the appropriate places.} \\
15x^3 - 15x^3 &\stackrel{?}{=} 0 \\
0 &= 0 && \text{This is true.}
\end{aligned}
$$

So, $y = 5x^3$ *is* a solution to the given differential equation.

12. **Solve the IVP (initial value problem) $1 = x^3 - y'(x)$ if $y(2) = 13$.**

We begin by isolating $y'(x)$. This gives $y'(x) = -1 + x^3$

Next we find the antiderivative of $y'(x)$: $y(x) = -x + \frac{x^4}{4} + C$.

Now we plug in the 2 and the 13 to find the value of C.

$$13 = -2 + \frac{2^4}{4} + C$$
$$13 = -2 + 4 + C$$
$$11 = C$$

So, the solution to this IVP is $y(x) = -x + \frac{x^4}{4} + 11$.