Math 105: Review for Exam I - Solutions

1. Let \( f(x) = 3 + \sqrt{x + 5} \).
   (a) What is the natural domain of \( f \)? \([-5, \infty)\), which means all reals greater than or equal to 5
   (b) What is the range of \( f \)? \([3, \infty)\), which means all reals greater than or equal to 3

2. For the graph of \( f \) shown, answer the following.
   (a) Evaluate the following.
      i. \( f'(-2) = 0 \)
      ii. \( f(3) = 2 \)
      iii. \( \lim_{x \to 3^-} f(x) = -2 \)
      iv. \( \lim_{x \to 3^+} f(x) = 3 \)
      v. \( \lim_{x \to 3} f(x) \) does not exist
      vi. \( \lim_{x \to 2} f(x) = 0 \)
   (b) Where is \( f \) discontinuous? at \( x = 3 \)
   (c) Where does \( f' \) fail to exist?
      at \( x = -3, -1, 0, 3 \)

3. Let \( f(x) = 3x^2 - 2x \).
   (a) Compute the average rate of change of \( f \) on the interval \([2, 2.1]\).
      \[
      \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.03 - 8}{0.1} = 10.3
      \]
   (b) Using the limit definition of the derivative, find \( f'(x) \).
      \[
      f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
      \]
      provided this limit exists
      \[
      = \lim_{h \to 0} \frac{3(x + h)^2 - 2(x + h) - (3x^2 - 2x)}{h}
      = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h}
      = \lim_{h \to 0} \frac{6x + 3h - 2}{h}
      = \lim_{h \to 0} (6x + 3h - 2)
      = 6x - 2
      \]
   (c) Find the equation of the tangent line to \( f \) at \( x = 2 \).
      We want \( y = mx + b \). \( m = f'(2) = 6 \cdot 2 - 2 = 10 \), so \( y = 10x + b \).
      When \( x = 2 \), \( y = f(2) = 3 \cdot 2^2 - 2 \cdot 2 = 8 \).
      Thus, \( 8 = 10 \cdot 2 + b \), so \( b = -12 \) and we have \( y = 10x - 12 \).
   (d) How would the derivative of \( g(x) = f(x) + 5 \) compare to \( f'(x) \)?
      The graph of \( y = f(x) + 5 \) is the graph of \( y = f(x) \) shifted vertically by 5 units, but this has no effect on the slope of the graph, so \( g'(x) = f'(x) \).
(e) **How would the derivative of** \( h(x) = 5f(x) \) **compare to** \( f'(x) \)?

The graph of \( y = 5f(x) \) is the graph of \( y = f(x) \) stretched vertically by a factor of 5; this also results in slopes that are 5 times greater at any given \( x \)-value. Thus, \( h'(x) = 5f'(x) \).

Note that we get the same result by considering our derivative rule \( \frac{d}{dx}[kf(x)] = kf'(x) \) where \( k = 5 \).

4. Fill in the table showing the graphical relationships between \( f \), \( f' \), and \( f'' \).

<table>
<thead>
<tr>
<th>( f' ) positive</th>
<th>negative</th>
<th>increasing</th>
<th>decreasing</th>
<th>concave up</th>
<th>concave down</th>
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<tbody>
<tr>
<td>( f'' )</td>
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<td>X</td>
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</table>

5. Given the graph of \( f \), sketch a graph of \( f' \) and a graph of \( F \), an antiderivative of \( f \) such that \( F(0) = -2 \).

Note: The concave up portion in the middle of the graph of \( f \) is a perfect parabola, so its derivative \( (f') \) is linear; since you don’t know the equation for \( f \), your graph of \( f' \) may be concave up/down there.
6. Shown below is a graph of \( f' \) on its entire domain. The graph is NOT \( f \).

At which \( x \)-value(s)

(a) does \( f \) have a stationary point? \( c, e, j \)
(b) \( f \) decreasing? \([a, c) \cup (e, k]\)
(c) \( f' \) increasing? \([a, d) \cup (h, j]\)
(d) \( f' \) decreasing? \((d, h) \cup (j, k]\)
(e) \( f \) concave up? \([a, d) \cup (h, j]\)
(f) \( f \) concave down? \((d, h) \cup (j, k]\)

On what interval(s) is

(a) \( f \) increasing? \((c, e)\)

7. Suppose that \( T(t) \) gives the temperature in Lewiston as a function of time. In each of the following situations, determine if the signs of \( T \), \( T' \), and \( T'' \) are positive, negative, zero, or unknown.

(a) The temperature is 60 degrees and falling steadily.
   The temperature is 60, so we know \( T \) is positive.
   The temperature is falling, so we know \( T' \) is negative.
   The temperature is falling steadily, so we know the graph is linear, and \( T'' \) is zero.

(b) The temperature is rising more and more slowly.
   We don’t know whether the temperature is above or below zero, so the sign of \( T \) is unknown.
   The temperature is rising, so we know \( T' \) is positive.
   The temperature is rising more and more slowly, so we know the graph of \( T \) is concave down, and \( T'' \) is negative.

(c) The temperature is \(-5\) degrees and rising.
   The temperature is \(-5\), so we know \( T \) is negative.
   The temperature is rising, so we know \( T' \) is positive.
   We don’t know the concavity of the graph of \( T \), so the sign of \( T'' \) is unknown.

8. An object has vertical velocity \( v(t) = t^2 - 5t + 4 \) feet per second on the interval \([0, 5]\). A positive velocity indicates the object is ascending, and a negative velocity indicates it is descending. At time \( t = 0 \), the object is 50 feet above ground.

(a) When is the object ascending? When is it descending?
   \( v(t) = (t - 1)(t - 4) = 0 \) when \( t = 1 \) and \( t = 4 \).
   On the intervals \([0, 1]\) and \((4, 5]\), \( v \) is positive, so the object is ascending during these times.
On the interval \((1, 4)\), \(v\) is negative, so the object is descending during this time.

(b) **When is the object’s acceleration most positive?**
   The acceleration is \(a(t) = 2t - 5\). This is an increasing function, so it is most positive at the right endpoint of the interval, or at \(t = 5\).

(c) **What is the greatest height the object reaches?**
   From part (a), we know that the object is rising until \(t = 1\) and again until \(t = 5\), so the greatest height must occur at one of those times.

   We can find the height by taking the antiderivative of \(v(t)\), which gives \(h(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 4t + C\).

   Since we know \(h(0) = 50\), we see that \(C = 50\).

   Now we compare the heights at \(t = 1\) and \(t = 5\).

   \(h(1) = 311/6\) and \(h(5) = 295/6\), so the maximum height is \(311/6 = 51.83\) feet.

9. Find the derivatives of the following.
   
   (a) \(y = 2 + 3x + x^4 + 5x^6\)
   \[y' = 3 + 4x^3 + 30x^5\]

   (b) \(y = \sqrt{x} + \frac{1}{x^6} + \frac{x}{6} + \frac{x}{6} + 6^{1/2} + \sqrt{6x^{1/6}}\)
   First, rewrite \(y\) to make it easier to apply our derivative rules:
   \[y = x^{1/6} + x^{-6} + \frac{1}{6} \cdot x + 6x^{-1} + \frac{\pi}{6} + 6^{1/2} + 6^{1/2}x^{1/12}\]
   Note that above \(\sqrt{6x^{1/6}} = \sqrt{6} \sqrt{x^{1/6}} = 6^{1/2}(x^{1/6})^{1/2} = 6^{1/2}x^{1/12}\).
   \[y' = \frac{1}{6} x^{-5/6} + (-6)x^{-7} + \frac{1}{6} + (6)(-1)(x^{-2}) + 0 + 0 + 6^{1/2} \frac{1}{12} x^{-11/12}\]
   To clean this up, we use exponent rules: \(x^{-a} = \frac{1}{x^a}\) and \(x^{a/b} = \sqrt[b]{x^a}\)
   \[y' = \frac{1}{6 \sqrt[6]{x^5}} - \frac{6}{x^7} + \frac{1}{6} - \frac{6}{x^2} + \frac{\sqrt[6]{6}}{12 \sqrt[12]{x^{11}}}\]

10. Find antiderivatives of the following.
    
    (a) \(y = \pi + 3x^2\)
    antiderivative = \(\pi x + x^3 + C\)

    (b) \(y = 4x^5 - \frac{1}{x^6}\) = \(4x^5 - x^{-6}\)
    antiderivative = \(\frac{4x^6}{6} - \frac{x^{-5}}{-5} + C = \frac{2x^6}{3} + \frac{1}{5x^5} + C\)

11. **Is \(y = 5x^3\) a solution to the differential equation \(xy' - 3y = 0\)?**
    The question asks whether, when we plug in \(y\) and \(y'\), \(xy' - 3y\) will equal 0.
    We are given \(y = 5x^3\), so \(y' = 15x^2\).
    \[xy' - 3y = 0\]
    \[x \cdot 15x^2 - 3 \cdot 5x^3 = 0\]
    Substitute \(y\) and \(y'\) in the appropriate places.
    \[15x^3 - 15x^3 = 0\]
    \[0 = 0\]
    This is true.

    So, \(y = 5x^3\) is a solution to the given differential equation.
12. **Solve the IVP (initial value problem)** \( 1 = x^3 - y'(x) \) if \( y(2) = 13 \).

We begin by isolating \( y'(x) \). This gives \( y'(x) = -1 + x^3 \).

Next we find the antiderivative of \( y'(x) \): \( y(x) = -x + \frac{x^4}{4} + C \).

Now we plug in the 2 and the 13 to find the value of \( C \).

\[
\begin{align*}
13 &= -2 + \frac{2^4}{4} + C \\
13 &= -2 + 4 + C \\
11 &= C
\end{align*}
\]

So, the solution to this IVP is \( y(x) = -x + \frac{x^4}{4} + 11 \).