

Math 105: Review for Exam I - Solutions

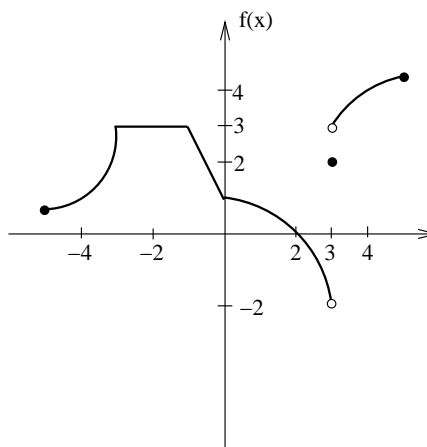
1. Let  $f(x) = 3 + \sqrt{x+5}$ .

- (a) What is the natural domain of  $f$ ?  $[-5, \infty)$ , which means all reals greater than or equal to 5
- (b) What is the range of  $f$ ?  $[3, \infty)$ , which means all reals greater than or equal to 3

2. For the graph of  $f$  shown, answer the following.

(a) Evaluate the following.

- i.  $f'(-2) = 0$
- ii.  $f(3) = 2$
- iii.  $\lim_{x \rightarrow 3^-} f(x) = -2$
- iv.  $\lim_{x \rightarrow 3^+} f(x) = 3$
- v.  $\lim_{x \rightarrow 3} f(x)$  does not exist
- vi.  $\lim_{x \rightarrow 2} f(x) = 0$



- (b) Where is  $f$  discontinuous? at  $x = 3$
- (c) Where does  $f'$  fail to exist?  
at  $x = -3, -1, 0, 3$

3. Let  $f(x) = 3x^2 - 2x$ .

(a) Compute the average rate of change of  $f$  on the interval  $[2, 2.1]$ .

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.03 - 8}{0.1} = 10.3$$

(b) Using the limit definition of the derivative, find  $f'(x)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h - 2)}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h - 2) \\ &= 6x - 2 \end{aligned}$$

(c) Find the equation of the tangent line to  $f$  at  $x = 2$ .

We want  $y = mx + b$ .  $m = f'(2) = 6 \cdot 2 - 2 = 10$ , so  $y = 10x + b$ .

When  $x = 2$ ,  $y = f(2) = 3 \cdot 2^2 - 2 \cdot 2 = 8$ .

Thus,  $8 = 10 \cdot 2 + b$ , so  $b = -12$  and we have  $y = 10x - 12$ .

(d) How would the derivative of  $g(x) = f(x) + 5$  compare to  $f'(x)$ ?

The graph of  $y = f(x) + 5$  is the graph of  $y = f(x)$  shifted vertically by 5 units, but this has no effect on the slope of the graph, so  $g'(x) = f'(x)$ .

(e) How would the derivative of  $h(x) = 5f(x)$  compare to  $f'(x)$ ?

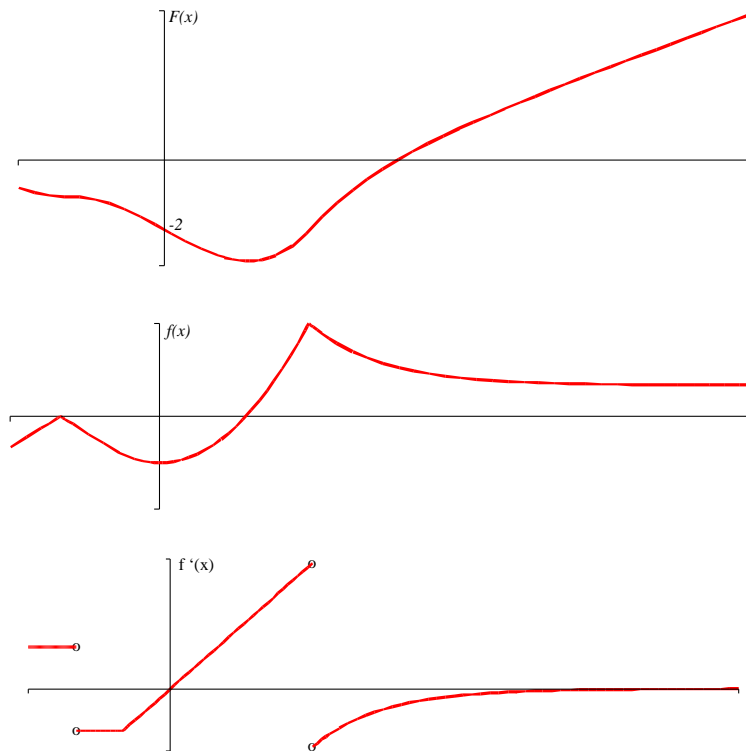
The graph of  $y = 5f(x)$  is the graph of  $y = f(x)$  stretched vertically by a factor of 5; this also results in slopes that are 5 times greater at any given  $x$ -value. Thus,  $h'(x) = 5f'(x)$ .

Note that we get the same result by considering our derivative rule  $\frac{d}{dx} [kf(x)] = kf'(x)$  where  $k = 5$ .

4. Fill in the table showing the graphical relationships between  $f$ ,  $f'$ , and  $f''$ .

|       |          |          |            |            |            |              |
|-------|----------|----------|------------|------------|------------|--------------|
| $f$   | positive | negative | increasing | decreasing | concave up | concave down |
| $f'$  | X        | X        | positive   | negative   | increasing | decreasing   |
| $f''$ | X        | X        | X          | X          | positive   | negative     |

5. Given the graph of  $f$ , sketch a graph of  $f'$  and a graph of  $F$ , an antiderivative of  $f$  such that  $F(0) = -2$ .



Note: The concave up portion in the middle of the graph of  $f$  is a perfect parabola, so its derivative ( $f'$ ) is linear; since you don't know the equation for  $f$ , your graph of  $f'$  may be concave up/down there.

6. Shown below is a graph of  $f'$  on its entire domain. The graph is NOT  $f$ .

At which  $x$ -value(s)

(a) does  $f$  have a stationary point?  $c, e, j$

(b) does  $f$  have a local max?  $e$

(c) does  $f$  have a local min?  $c$

(d) does  $f'$  have a stationary point?  $d, h, j$

(e) does  $f'$  have a local max?  $d, j$

(f) does  $f'$  have a local min?  $h$

(g) is  $f$  greatest?  $a$

(h) is  $f$  least?  $k$

(i) is  $f'$  greatest?  $d$

(j) is  $f'$  least?  $a$

(k) is  $f''$  greatest?  $b$

(l) is  $f''$  least?  $f$

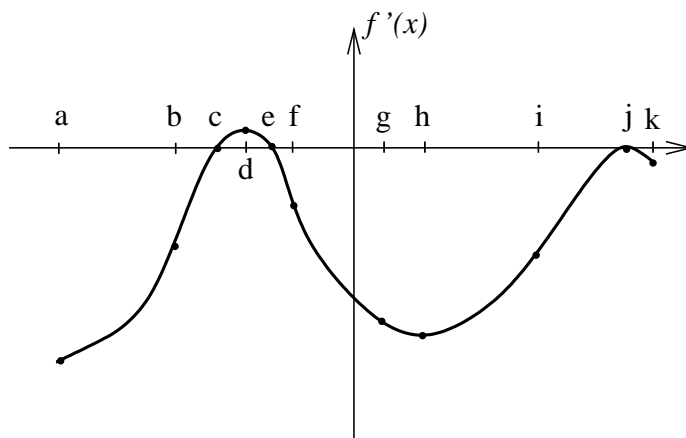
(b)  $f$  decreasing?  $[a, c) \cup (e, k]$

(c)  $f'$  increasing?  $[a, d) \cup (h, j)$

(d)  $f'$  decreasing?  $(d, h) \cup (j, k]$

(e)  $f$  concave up?  $[a, d) \cup (h, j)$

(f)  $f$  concave down?  $(d, h) \cup (j, k]$



On what interval(s) is

(a)  $f$  increasing?  $(c, e)$

7. Suppose that  $T(t)$  gives the temperature in Lewiston as a function of time. In each of the following situations, determine if the signs of  $T$ ,  $T'$ , and  $T''$  are positive, negative, zero, or unknown.

(a) **The temperature is 60 degrees and falling steadily.**

The temperature is 60, so we know  $T$  is positive.

The temperature is *falling*, so we know  $T'$  is negative.

The temperature is falling *steadily*, so we know the graph is linear, and  $T''$  is zero.

(b) **The temperature is rising more and more slowly.**

We don't know whether the temperature is above or below zero, so the sign of  $T$  is unknown.

The temperature is *rising*, so we know  $T'$  is positive.

The temperature is rising *more and more slowly*, so we know the graph of  $T$  is concave down, and  $T''$  is negative.

(c) **The temperature is  $-5$  degrees and rising.**

The temperature is  $-5$ , so we know  $T$  is negative.

The temperature is *rising*, so we know  $T'$  is positive.

We don't know the concavity of the graph of  $T$ , so the sign of  $T''$  is unknown.

8. An object has vertical velocity  $v(t) = t^2 - 5t + 4$  feet per second on the interval  $[0, 5]$ . A positive velocity indicates the object is ascending, and a negative velocity indicates it is descending. At time  $t = 0$ , the object is 50 feet above ground.

(a) **When is the object ascending? When is it descending?**

$v(t) = (t - 1)(t - 4) = 0$  when  $t = 1$  and  $t = 4$ .

On the intervals  $[0, 1)$  and  $(4, 5]$ ,  $v$  is positive, so the object is ascending during these times.

On the interval  $(1, 4)$ ,  $v$  is negative, so the object is descending during this time.

- (b) **When is the object's acceleration most positive?**

The acceleration is  $v'(t) = 2t - 5$ . This is an increasing function, so it is most positive at the right endpoint of the interval, or at  $t = 5$ .

- (c) **What is the greatest height the object reaches?**

From part (a), we know that the object is rising until  $t = 1$  and again until  $t = 5$ , so the greatest height must occur at one of those times.

We can find the height by taking the antiderivative of  $v(t)$ , which gives  $h(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 4t + C$ .

Since we know  $h(0) = 50$ , we see that  $C = 50$ .

Now we compare the heights at  $t = 1$  and  $t = 5$ .

$h(1) = 311/6$  and  $h(5) = 295/6$ , so the maximum height is  $311/6 = 51.8\bar{3}$  feet.

9. **Find the derivatives of the following.**

(a)  $y = 2 + 3x + x^4 + 5x^6$

$$y' = 3 + 4x^3 + 30x^5$$

(b)  $y = \sqrt[6]{x} + \frac{1}{x^6} + \frac{x}{6} + \frac{6}{x} + \frac{\pi}{6} + 6^{1/2} + \sqrt{6x^{1/6}}$

First, rewrite  $y$  to make it easier to apply our derivative rules:

$$y = x^{1/6} + x^{-6} + \frac{1}{6} \cdot x + 6x^{-1} + \frac{\pi}{6} + 6^{1/2} + 6^{1/2}x^{1/12}$$

$$\text{Note that above } \sqrt{6x^{1/6}} = \sqrt{6}\sqrt{x^{1/6}} = 6^{1/2}(x^{1/6})^{1/2} = 6^{1/2}x^{1/12}.$$

$$y' = \frac{1}{6}x^{-5/6} + (-6)x^{-7} + \frac{1}{6} + (6)(-1)(x^{-2}) + 0 + 0 + 6^{1/2} \frac{1}{12}x^{-11/12}$$

To clean this up, we use exponent rules:  $x^{-a} = \frac{1}{x^a}$  and  $x^{a/b} = \sqrt[b]{x^a}$

$$y' = \frac{1}{6\sqrt[6]{x^5}} - \frac{6}{x^7} + \frac{1}{6} - \frac{6}{x^2} + \frac{\sqrt{6}}{12\sqrt[12]{x^{11}}}$$

10. **Find antiderivatives of the following.**

(a)  $y = \pi + 3x^2$

$$\text{antiderivative} = \pi x + x^3 + C$$

(b)  $y = 4x^5 - \frac{1}{x^6} = 4x^5 - x^{-6}$

$$\text{antiderivative} = \frac{4x^6}{6} - \frac{x^{-5}}{-5} + C = \frac{2x^6}{3} + \frac{1}{5x^5} + C$$

11. **Is  $y = 5x^3$  a solution to the differential equation  $xy' - 3y = 0$ ?**

The question asks whether, when we plug in  $y$  and  $y'$ ,  $xy' - 3y$  will equal 0.

We are given  $y = 5x^3$ , so  $y' = 15x^2$ .

$$xy' - 3y \stackrel{?}{=} 0$$

$$x \cdot 15x^2 - 3 \cdot 5x^3 \stackrel{?}{=} 0 \quad \text{Substitute } y \text{ and } y' \text{ in the appropriate places.}$$

$$15x^3 - 15x^3 \stackrel{?}{=} 0$$

$$0 = 0 \quad \text{This is true.}$$

So,  $y = 5x^3$  is a solution to the given differential equation.

12. Solve the IVP (initial value problem)  $1 = x^3 - y'(x)$  if  $y(2) = 13$ .

We begin by isolating  $y'(x)$ . This gives  $y'(x) = -1 + x^3$

Next we find the antiderivative of  $y'(x)$ :  $y(x) = -x + \frac{x^4}{4} + C$ .

Now we plug in the 2 and the 13 to find the value of  $C$ .

$$13 = -2 + \frac{2^4}{4} + C$$

$$13 = -2 + 4 + C$$

$$11 = C$$

So, the solution to this IVP is  $y(x) = -x + \frac{x^4}{4} + 11$ .