1. The slope field for the differential equation \( \frac{dy}{dt} = t^2 - y \) is drawn above. Sketch the solution \( y_1(t) \) through the point \((-1.5, 0)\). According to your plot, what's \( y_1(1) \)? Circle the corresponding point on your plot.

\[
y_1(1) \approx 0.4
\]

2. Consider the initial value problem

\[
\begin{align*}
\frac{dy}{dt} &= t^2 - y \\
y(-1) &= 2
\end{align*}
\]

2a. Find the value of \( C \) such that \( y_2(t) = t^2 - 2t + 2 + Ce^{-t} \) is the solution of this IVP. Show all your work.

\[
C = -\frac{3}{e}
\]

2b. Use your calculator to find the first four Euler approximations to this solution, using a step size of \( \Delta t = 0.25 \). Organize your results in a table and plot the points on the slope field. Write your answers to three decimal places.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>-0.75</td>
<td>1.75</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.453</td>
</tr>
<tr>
<td>-0.25</td>
<td>1.152</td>
</tr>
<tr>
<td>0</td>
<td>0.880</td>
</tr>
</tbody>
</table>

This pair is given. The next four are the approximations. These are plotted above and joined by straight line segments (just to help visualize the solution).
3. Suppose the change in position $f(t)$ in feet per second of an object is measured and the results put into a table as follows:

<table>
<thead>
<tr>
<th>time:</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
<th>2.00</th>
<th>2.25</th>
<th>2.50</th>
<th>2.75</th>
<th>3.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$:</td>
<td>9.8</td>
<td>9.4</td>
<td>8.6</td>
<td>7.4</td>
<td>5.4</td>
<td>2.6</td>
<td>-1.0</td>
<td>-5.4</td>
<td>-10.6</td>
<td>-16.6</td>
<td>-23.4</td>
</tr>
</tbody>
</table>

a. What does the integral \( \int_{1.00}^{2.50} f(t) \, dt \) represent? (Watch your limits!)

The net change in position, in feet, of the object, from \( t = 1 \) to \( t = 2.5 \).

b. Find each of the following estimates of the integral in (a) using only the information in the table. If the table doesn't supply the information needed, explain why not.

\[
T_3 = \frac{L_3 + R_3}{2} = \frac{1}{2} \left( 0.5 \left( 8.6 + 5.4 - 1.0 \right) + 0.5 \left( 5.4 - 1.0 - 10.6 \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \cdot 13 + \frac{1}{2} \cdot (-6.2) \right) = \frac{1}{2} \left( 6.5 - 3.1 \right) = 1.7
\]

We do not have values of \( f(t) \) at the midpoints of 5 equal subdivisions of \([1.00, 2.50]\), i.e., at \( t = 1.15, 1.45, 1.75, 2.05, \) and \( 2.35 \).

\[
S_6 = \frac{2M_3 + T_3}{3} = 2 \left( \frac{0.5 \left( 9.4 + 2.6 - 5.4 \right) + 1.7}{3} \right)
\]

\[
= \frac{2 \left( 2.3 \right) + 1.7}{3} = \frac{6.3}{3} = 2.1
\]

c. Is \( R_6 \) an under- or over-approximation of the integral in (a)? Explain!

\( f(t) \) is a decreasing function on \([1, 2.50]\), so the right-hand sums will be under approximates.

![Diagram showing concavity down]

[Note: concavity doesn't matter.]

[d. Is \( M_3 \) an under- or over-approximation of the integral in (a)? Explain!]

\( f(t) \) is concave down on \([1, 2.50]\) because its values are changing at a decreasing rate [falling faster & faster, so \( M_3 \) is an over approx, since tangent lines are above the curve.]

[Note: increasing & decreasing don't matter.]
Use the method of substitution to find each of the following integrals. You may find the table useful, if so, tell which formula you used by its number.

4. Use substitution to find the following integral. Show the limits of integration as they appear on the (new) integral once the substitution has been made. \( \int_{e}^{2} (\ln x)^{3} / x \, dx \)

\[
\text{let } u = \ln x \\
\text{Then } du = \frac{1}{x} \, dx. \text{ Also, if } x = e \text{ then } u = \ln(e) = 1 \\
\text{and if } x = e^2 \text{ then } u = \ln(e^2) = 2.
\]

the integral in terms of \( u \) is then

\[
\int_{1}^{2} u^3 \, du = \left. \frac{u^4}{4} \right|_{1}^{2} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}
\]

New (post-substitution) integral & its limits:

\[
\int_{1}^{2} u^3 \, du \\
\text{final result: } 3^{3/4}
\]

5. \( \int \frac{e^{5/x} \sin 5/x}{4x^2} \, dx \)

at least two ways to do it:

- \( \text{let } u = 5/x = 5x^{-1} \). So \\
  \( du = -5x^{-2} \, dx \) \\
  The integral becomes

\[
\int \frac{e^{5/x} \sin 5/x}{4x^2} \, dx = \int \frac{-e^u \sin u}{20} \, du
\]

- \( \text{let } u = 1/x \); \( du = -1/x^2 \, dx \); integral is:

\[
-\frac{1}{4} \int e^{5u} \sin 5u \, du
\]

\[
= -\frac{1}{4} \cdot \frac{e^{5u}}{50} (5 \sin 5u - 5 \cos 5u) + C
\]

(\( a = b = 5 \); this is \( 5^2 + 5^2 \))

\[
= -\frac{1}{4} \cdot \frac{5^5}{10} (\sin 5x - \cos 5x) + C
\]

(some 5's have been cancelled)

\[
= \text{the same as this}
\]
6. Let \( I = \int_{0.25}^{1} x^5 - 4x^3 \). Use theorem 3 from section 6.2 to find the least \( n \) such that \( |I - M_n| \leq 0.01 \). In your work find (to the tenth's place) the "best" possible \( K_2 \) as we've done in class. Show all your work.

That theorem says that \( |I - M_n| \leq \frac{K_2 (b-a)^3}{24n^2} \) where \( |f''(x)| \leq K_2 \) for all \( x \) in \([a, b]\).

For \( f(x) = x^5 - 4x^3 \),

\[ f'(x) = 5x^4 - 12x^2 \]

and

\[ f''(x) = 20x^3 - 24x. \]

A graph of \( f''(x) \) on \([0.25, 1.0]\) shows we should choose \( K_2 = 10.2 \).

we need to choose \( n \) so that \( \frac{K_2 (b-a)^3}{24n^2} \leq 0.01 \).

So \( \frac{10.2 (1-0.25)^3}{24n^2} \leq 0.01 \)

\( \Rightarrow \frac{10.2 (0.75)^3}{24 \times 0.01} \leq n^2 \) \( \Rightarrow 17.929 \leq n^2 \) \( \Rightarrow 4.23 \leq n \)

\( \text{The least integer } n \text{ is therefore } n = 5 \).

7. A water tank underground has the shape shown here. The top of the tank is 5 feet under the ground, and the water is 2 feet deep in the center. What work is done to empty this tank, if the water is pumped up to ground level? Set up but don't evaluate the integral.

The volume of a thin slice at height \( y \) is \( \pi r^2 \, dy \). This radius needs to be in terms of \( y \). We can write \( r \) as the \( x \) coord. of this point.

Since \( y = e^x - 1 \), \( x = \ln(y+1) \)

The force on the slice is now \( 62.4 \pi (\ln(y+1))^2 y \) lbs, and the distance it travels is \((10-y)\) feet. Finally, "slices" of water occupy the tank from 0 to 2.

Thus, the integral is

\[ \int_{0}^{2} 62.4 \pi (\ln(y+1))^2 (10-y) \, dy \] (foot pounds)
8. Let \( R \) be the region between the two curves shown here. In each part below, set up, but don’t evaluate, an integral representing the given quantity:

8a. The area of \( R \).

\[
\int_{-1}^{3} (3-x^2) - (-2x) \, dx
\]

(as the length of this line segment
(at an arbitrary \( x \) between -1 and 3)
is \( (\text{top y-coord}) - (\text{bottom y-coord}) = (3-x^2) - (-2x) \)

8b. The volume of the solid of revolution obtained by revolving this region about the line \( y = 5 \).

\[
\int_{-1}^{3} \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 \, dx
\]

\[
= \int_{-1}^{3} \pi (5-2x)^2 - \pi (5-(3-x^2))^2 \, dx
\]

\[
= \int_{-1}^{3} \pi (5+2x)^2 - \pi (2+x^2)^2 \, dx
\]