NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

*Advice:* DON’T spend too much time on a single problem.

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1. Let $P$ be the parallelogram in $\mathbb{R}^3$ with vertices $(2, 1), (-1, 4), (6, 3)$ and $(3, 6)$.

(i) Find the area of $P$.

Let $a = (-1, 4) - (2, 1) = (-3, 3)$ and $b = (6, 3) - (2, 1) = (4, 2)$. Then the area of $P$ is equal to $||a \times b|| = 18$.

(ii) Suppose $T(x_1, x_2) = (5x_1 + 4x_2, 5x_1 + 3x_2)$. Find the associated matrix $A$ such that $T(x) = Ax$ where $x = (x_1, x_2)$.

The associated matrix is

$$A = \begin{pmatrix} 5 & 4 \\ 5 & 3 \end{pmatrix}$$

(iii) What are the vertices of the image $T(P)$?

The vertices of $T(P)$ are $(11, 7), (14, 13), (42, 39), and (39, 33)$.

(iv) What is the area of $T(P)$?

Area of $T(P)$ is given by

$$| \det A | \cdot \text{area of } P = |(5)(3) - (5)(4)| \cdot 18 = 90.$$ 

(v) What is the angle of the parallelogram $P$ at the vertex $(6, 3)$?

(You may express it in terms of inverse trig function.)

Since $||a \times b|| = 18 = (||a||)(||b||) \sin \theta$. It follows that $\theta = \arcsin \left( \frac{3}{\sqrt{10}} \right)$.
2. Let \( \mathbf{a} = \mathbf{i} + \mathbf{j} - 6\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \) and \( \mathbf{c} = -\mathbf{i} + 2\mathbf{k}. \)

(7 pts) (i) Find a vector that is perpendicular to the plane spanned by \( \mathbf{a} \) and \( \mathbf{b}. \)

The cross product \( \mathbf{a} \times \mathbf{b} \) is orthogonal to the plane containing \( \mathbf{a} \) and \( \mathbf{b}. \)

This vector is given by

\[
\mathbf{a} \times \mathbf{b} = \text{det} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -6 \\ 4 & 3 & 1 \end{pmatrix} = 19\mathbf{i} - 25\mathbf{j} - \mathbf{k}.
\]

(7 pts) (ii) What is the volume of the parallelepiped formed by \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c}. \)

The volume is given by

\[
|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |(-1, 0, 2) \cdot (19, -25, -1)| = 21.
\]

(6 pts) (iii) Find \( \text{proj}_\mathbf{c} \mathbf{a}. \)

Since \( \mathbf{a} \cdot \mathbf{c} = (1)(-1) + (1)(0) + (-6)(2) = -13 \) and \( ||\mathbf{c}||^2 = (-1)^2 + (2)^2 = 5, \)

it follows that

\[
\text{proj}_\mathbf{c} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{c}}{||\mathbf{c}||^2} \mathbf{c} = \frac{13}{5} \mathbf{i} - \frac{26}{5} \mathbf{k}.
\]
3. Let \( f(x, y) = x^2 + 2y^2 - 1 \). (5 pts) (i) Sketch the level curves of \( f(x, y) = 3 \), \( f(x, y) = -1 \), and \( f(x, y) = -5 \).

(5 pts) (iii) What are the cylindrical coordinates of the point \((1, 2, 8)\)?

\[
\text{In cylindrical coordinates, we have } r = \sqrt{x^2 + y^2} = \sqrt{5} \text{ and } \tan \theta = y/x \text{ so that } \theta = \arctan 2. \text{ The point } (1, 2, 8) \text{ becomes } (\sqrt{5}, \arctan 2, 8) \text{ in cylindrical coordinates.}
\]

(5 pts) (iv) Write the equation \( z = x^2 + 2y^2 - 1 \) in spherical coordinates.

\[
\text{In spherical coordinates, } x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, \text{ and } z = \rho \cos \phi. \text{ Thus the equation in spherical coordinates becomes } \rho \cos \phi = (\rho \sin \phi)^2 \cos^2 \theta + 2\rho^2 \sin^2 \phi \sin^2 \theta - 1.
\]
4. The velocity of a particle in $\mathbb{R}^3$ is given by the parametrization  
\[ v(t) = i + (1 + t)j + \cos tk. \]

(10 pts) (i) Find the position $r(t)$ of the particle with the initial point $r(0) = i + k$.  

Note that  
\[ r(t) = \int v(t) \, dt \]
\[ = (t + C_1)i + (t + \frac{t^2}{2} + C_2)j + (\sin t + C_3)k \]
where $C_1, C_2,$ and $C_3$ are constants. Since $r(0) = i + k$, it follows that $C_1 = 1, C_2 = 0,$ and $C_3 = 1$. Hence,  
\[ r(t) = (t + 1)i + (t + \frac{t^2}{2})j + (\sin t + 1)k. \]

(10 pts) (ii) Give an equation (in vector form) of the line tangent to the path of the particle at the point $r(\pi)$.  

The vector $v(\pi) = i + (1 + \pi)j - k$ is in the same direction as the desired tangent line. Hence, the tangent line to the path at $r(\pi)$ is given by  
\[ x(t) = r(\pi) + tv(\pi) \]
\[ = (\pi + 1)i + (\pi + \frac{\pi^2}{2})j + k + t[i + (1 + \pi)j - k] \]
5. (10 pts) (i) Consider the function

\[ f(x, y) = \frac{xy}{x^2 + y^2}. \]

Determine whether the origin is a removable discontinuity of \( f \). Justify your answer. [Hint: try approaching \((0, 0)\) from different directions]

To determine whether \((0, 0)\) is a removable discontinuity of \( f \), it suffices to determine whether the limit \( \lim_{(x,y) \to (0,0)} f(x, y) \) exists. Suppose we let \((x, y)\) approach \((0, 0)\) along the line \( y = mx \) for some \( m \neq 0 \). For any such \( m \), \( f(x, y) = \frac{mx}{(1+m^2)x^2} \). Thus for \( m \neq 0 \),

\[ \lim_{(x,y) \to (0,0)} f(x, y) = \frac{m}{1 + m^2} \]

which clearly varies as \( m \) varies so the limit does not exist and hence \((0, 0)\) is not removable.

(10 pts) (ii) Give an equation for the plane that is perpendicular to the line with parametric equations \( x = 3t - 5, y = 7 - 2t, z = 8 - t \) and that contains the point \((1, -1, 2)\). [Hint: rewrite the line in vector form \( \mathbf{x} = t \mathbf{m} + \mathbf{x}_0 \)]

In vector form, the given line is \( \mathbf{x} = t \mathbf{m} + \mathbf{x}_0 \) where \( \mathbf{m} = (3, -2, -1) \) and \( \mathbf{x}_0 = (-5, 7, 8) \). This means that the vector \((3, -2, -1)\) is a vector perpendicular to the desired plane, which contains the point \((1, -1, 2)\). If an arbitrary point on this plane is denoted by \((x, y, z)\) then the vector \((x - 1, y + 1, z - 2)\) must be perpendicular to the vector \((3, -2, -1)\), in other words, the desired plane is given by

\[ (x - 1, y + 1, z - 2) \cdot (3, -2, -1) = 0 \quad \text{or} \quad 3x - 2y - z = 3. \]