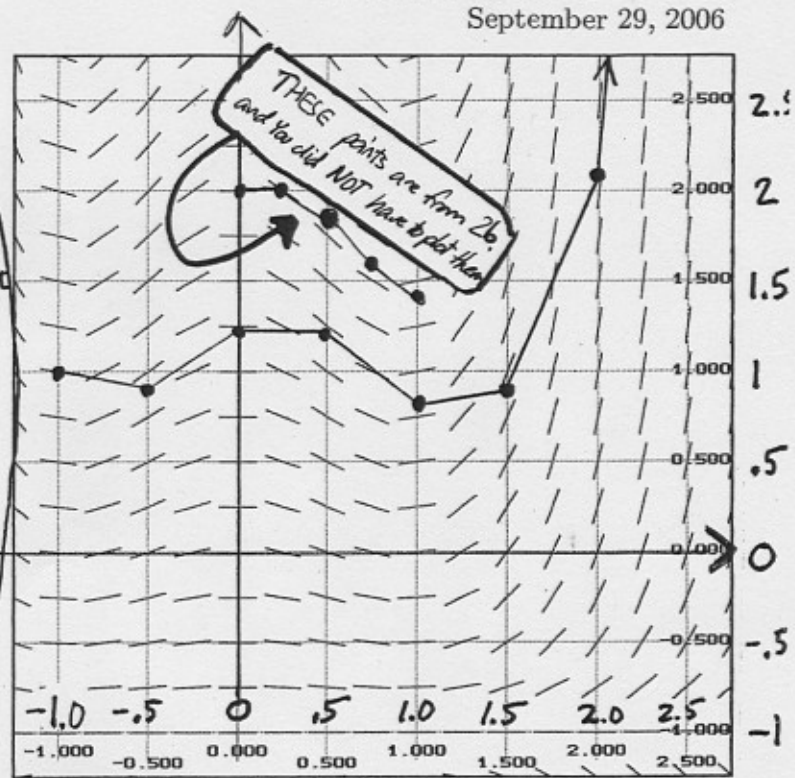


Remember:
Given any point (t_n, y_n) already "on the Euler method approximation", the next point is on the tangent line segment out of (t_n, y_n) and to the right. The line segment is an extension of the slope field element at (t_n, y_n) and it ends at the next approximation (t_{n+1}, y_{n+1}) .
[this written explanation is NOT required in your solution]



1. The slope field for the differential equation $dy/dt = (y+1)(t - \sin 2t)$ is drawn above. Using only a ruler and pencil (no calculator values), sketch the Euler method solution through the point $(-1, 1)$ as far as possible, using stepsize $h = 0.5$; your sketch will be a collection of big dots connected by line segments. (see sketch above)

2. Consider the initial value problem $\begin{cases} dy/dt = (y+1)(t - \sin 2t) \\ y(0) = 2 \end{cases}$.
(The solution is not the same as the one in problem 1 as it "starts" at a different point)
2a. Find the value of C such that $y(t) = Ce^{0.5(t^2 + \cos 2t)} - 1$ is the solution of this IVP. Show all your work. (You don't have to verify $y(t)$ "satisfies" the DE; just find C)

we need $y(0) = 2$

$$y(0) = 2 = Ce^{0.5(0^2 + \cos(2 \cdot 0))} - 1$$

$$2 = Ce^{0.5} - 1 \quad \therefore 3 = C\sqrt{e} \quad \rightarrow C = \frac{3}{\sqrt{e}} \approx 1.8196...$$

2b. Use your calculator to find the first four Euler approximations to the solution of the IVP in (2), using a step size of $h = 0.25$. Organize your results in a table, writing all results to four decimal places. Compare the exact value of $y(1)$ to its approximation in the table. How far apart are they?

(calculator steps: $0.25 \rightarrow H$, $2 \rightarrow Y$, $0 \rightarrow T$) set up then $\begin{cases} Y + (Y+1)(T - \sin(2T))H \rightarrow Y \\ T + H \rightarrow T \end{cases}$ repeat these to produce new pairs (t, y) in the approximation

the solution { your TABLE:

t	y
0	2 (starting point)
0.25	2
0.5	1.8279
0.75	1.5865
1.00	1.4265 = APPROX. of $y(1)$

EXACT VALUE is $1.8196(e^{0.5(1 + \cos 2)}) - 1 = 1.4364$
the difference is $1.4364 - 1.4265 \approx 0.0099$

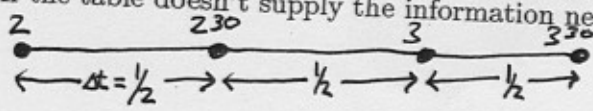
3. Suppose the rate of change in population $c(t)$ in thousands of organisms per hour is measured and the results put into a table as follows:

time:	1:30	1:45	2:00	2:15	2:30	2:45	3:00	3:15	3:30	3:45	4:00
$c(t)$:	1.1	1.2	1.4	1.8	2.6	4.2	7.4	13.8	26.6	52.2	103.4

a. What does the integral $\int_{2:00}^{3:30} c(t) dt$ represent? (Watch your limits!)

It's the net, or total, change in the population (in thousands of organisms) from $t = 2:00$ to $t = 3:30$.

b. Find each of the following estimates of the integral in (a) using only the information in the table. If the table doesn't supply the information needed, explain why not.

T_3  [note: Δt must be measured in HOURS; " $\Delta t = 30$ " is WRONG]

$$T_3 = \frac{L_3 + R_3}{2} = \frac{(1.4 + 2.6 + 7.4) \frac{1}{2} + (2.6 + 7.4 + 26.6) \frac{1}{2}}{2} = \frac{(11.4) \frac{1}{2} + (36.6) \frac{1}{2}}{2}$$

M_6

Can't do. we need values of $c(t)$ at the midpoints of the six intervals $[2, 2^{15}]$, $[2^{15}, 2^{30}]$, etc, and we are not given those (eg. we do not know $c(2:07:30)$)

$$= \frac{5.7 + 18.3}{2} = 24/2 = 12$$

S_6

$$S_6 = \frac{2M_3 + T_3}{2} \quad \text{Now, } M_3 = \frac{(1.8 + 4.2 + 13.8) \frac{1}{2}}{3} = \frac{(19.8) \frac{1}{2}}{3} = 9.9 \quad \text{so } S_6 = \frac{2(9.9) + 12}{3} = \frac{31.8}{3} = 10.6$$

c. Is T_3 an under- or over-approximation of the integral in (a)? Explain!

the rate of change of temperature function $c(t)$ is increasing at an increasing rate

ie. $c(t)$ is concave UP and we know T_n is an over approx:



d. If the population at 3:30 was 46 (thousand) what is your best estimate for the population at 2:00?

since S_6 is the best approx. we have to $\int_2^{3:30} c(t) dt$, and the integral represents the total change in pop, we assume the total change in pop. was about 10.6 (thousand). If the population was 46 (thousand) at the end of the time period, we expect it was $46 - 10.6 \approx 35.4$ (thousand) at the start (at 2:00)

Use the method of substitution to find each of the following integrals. You may find the table useful, if so, tell which formula you used by its number.

4. Use substitution to find the following integral. Show the limits of integration as they appear on the (new) integral once the substitution has been made. Leave your final answer in terms of e .

$$\int_{25}^{100} (e^{-\sqrt{x}}) / \sqrt{x} dx$$

let $u = -\sqrt{x} = -x^{1/2}$; then $du = -\frac{1}{2}x^{-1/2} = \frac{-1}{2\sqrt{x}} dx$ and $\begin{cases} x=25 \Rightarrow u = -\sqrt{25} = -5 \\ x=100 \Rightarrow u = -\sqrt{100} = -10 \end{cases}$

we need a $-\frac{1}{2}$ on the "inside" which we "pay for" with a -2 on the outside, and the \int becomes:

$$-2 \int_{25}^{100} e^{-\sqrt{x}} \frac{1}{2\sqrt{x}} dx = -2 \int_{-5}^{-10} e^u du = -2 e^u \Big|_{-5}^{-10} = -2 [e^{-10} - e^{-5}]$$

(Alternate: note if $u = \sqrt{x}$ then $du = \frac{1}{2}x^{-1/2} dx$, and the integral becomes

$$2 \int_5^{10} e^{-u} du \quad \text{BUT THIS is a little substitution problem: be careful: } \int e^{-u} du = -e^{-u},$$

and we get: $2(-e^{-u} \Big|_5^{10}) = 2(-e^{-10} - (-e^{-5})) = 2(e^{-5} - e^{-10})$

same.

5. $\int \frac{\cos(5 \ln x)}{x} \ln x dx$

METHOD 1:

let $u = 5 \ln x$; $du = \frac{5}{x} dx$;

\int becomes $\frac{1}{25} \int \cos(5 \ln x) \cdot (5 \ln x) \cdot \frac{5}{x} dx$

$= \frac{1}{25} \int (\cos u) (u) du$

use #47 with $a=1$ to get

$\frac{1}{25} (\cos u + u \sin u) + C$

$= \frac{1}{25} (\cos(5 \ln x) + (5 \ln x) \sin(5 \ln x)) + C$

METHOD 2:

let $u = \ln x$; $du = \frac{1}{x} dx$

\int becomes $\int \cos(5u) u du$

#47 with $a=5$ gives

$\frac{1}{25} (\cos 5u) + \frac{u}{5} (\sin 5u) + C$

$= \frac{1}{25} (\cos(5 \ln x)) + \frac{\ln x}{5} (\sin(5 \ln x)) + C$

note the two

answers are

identical

P
R
E
F
E
R
R
E
D

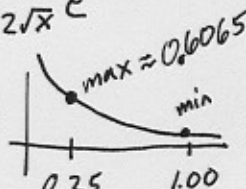
6. Let $I = \int_{0.25}^{1.0} e^{-\sqrt{x}} dx$. (This is not the same integral as in problem 4 and has nothing to do with it.) Use theorem 3 from section 6.2 to find the least n such that $|I - L_n| \leq 0.005$. In your work find (to three decimal places) the "best" possible K_1 as we've done in class. Show all your work. Then find L_n .

We know $|I - L_n| \leq \frac{K_1(b-a)^2}{2n}$ where $|f'(x)| < K_1$ on $[a, b]$.

so if we choose n such that $\frac{K_1(b-a)^2}{2n} \leq 0.005$ then since $|I - L_n|$ is smaller

Now, if $f(x) = e^{-\sqrt{x}} = e^{-(x^{1/2})}$ then $\frac{K_1(b-a)^2}{2n}$ it will also be ≤ 0.005

$f'(x) = -(x^{1/2})' e^{-x^{1/2}} = -\frac{1}{2}x^{-1/2} e^{-x^{1/2}} = -\frac{1}{2\sqrt{x}} e^{-\sqrt{x}}$

Graphing $|f'(x)| = \frac{1}{2\sqrt{x}} e^{-\sqrt{x}}$ we see: ; the max value of $|f'(x)|$ on $[0.25, 1.00]$ is ≈ 0.6065 !

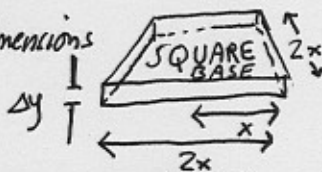
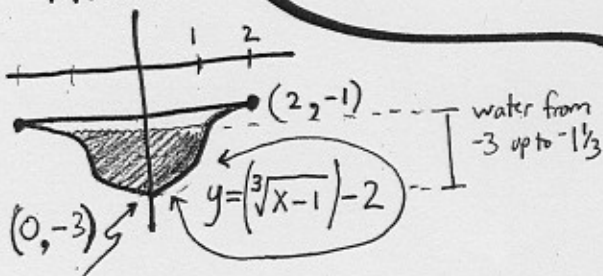
so take $K_1 = 0.607$. We need:

$\frac{0.607(1-0.25)^2}{2n} \leq 0.005$, or, $\frac{0.607 \cdot 0.5625}{2 \cdot 0.005} \leq n$ \rightarrow so $34.14 \leq n$. Since n is an integer we take $n=35$. Finally $L_{35} \approx 0.35065$.

7. A water tank under ground has the shape shown here. The front face of the tank is a plane, horizontal cross sections behind the front are squares, so the tank is longer from front to back the higher you go. The top of the tank is 1 foot under the ground, and the water is $1 \frac{2}{3}$ feet deep in the center at the front. What work is done to empty this tank, if the water is pumped up to 10' above ground level? Set up but don't evaluate the integral. Water weighs 62.4 lbs per ft^3 .

these pictures were given on the exam;

FRONT:



Its weight is $62.4 (2x)^2 \cdot dy$ & it travels $(10 - y)$ feet. We need x in terms of y ; $x = (y+2)^2 + 1$.

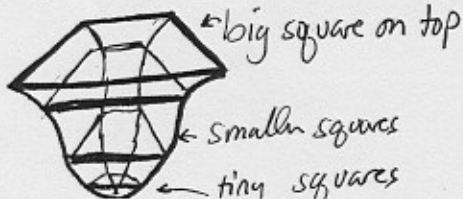
FINALLY: WORK =

$$\int_{y=-3}^{y=-1\frac{1}{3}} 62.4 (2((y+2)^2 + 1))^2 (10 - y) dy \quad (\text{foot-lbs})$$

$\downarrow \quad \downarrow$
 $(2x)^2$

Note that M_{100} gives 0.34807 which is probably a very good approx; the difference $|M_{100} - L_{35}|$ is 0.00258, which is less than 0.005....

PERSPECTIVE:



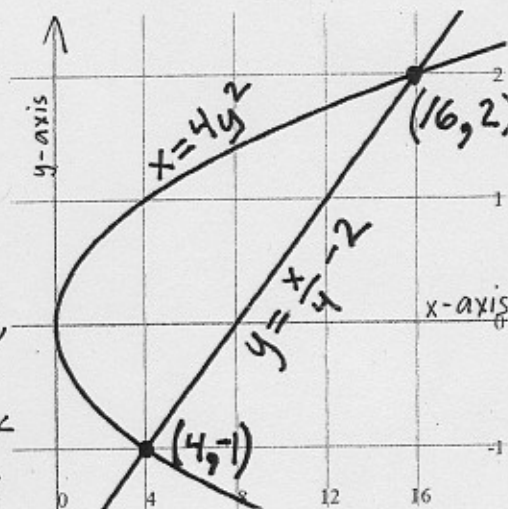
8. Let R be the region between the two curves shown here. In each part below, set up, but don't evaluate, an integral, or if necessary, a sum of two integrals, which represents the given quantity.

8a. The area of R , where the integral(s) is (are) in terms of dx .

Since the integral(s) are in terms of dx , approximating rectangles have width Δx and the length of their left sides is the difference between y -coordinates "on the top curve & bottom curve"

From $x=0$ to 4 , the top curve's & bottom curve's y coordinates are $y_{top} = \sqrt{x}/2$ and $y_{bottom} = -\sqrt{x}/2$; from $x=4$ to 16 , the top is the same but $y_{bottom} = x/4 - 2$. We require two integrals:

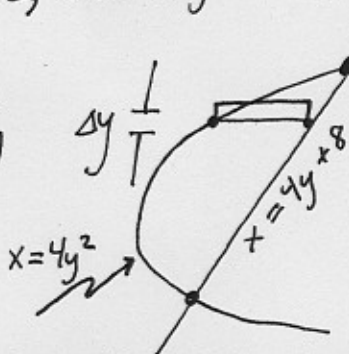
$$\int_0^4 \sqrt{x}/2 - (-\sqrt{x}/2) dx + \int_4^{16} \sqrt{x}/2 - (x/4 - 2) dx$$



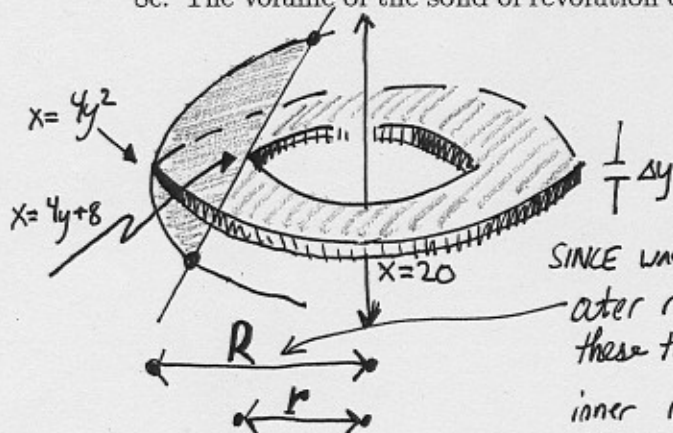
8b. Again, the area of R , where the integral(s) is (are) now in terms of dy .

Here the approximating rectangles have height Δy , and the length is the difference between x -coordinates "on the right most curve & left most curve". All the way up from $y=-1$ to $y=2$, that length is " $R-L$ " = $x_{right} - x_{left}$

So:
$$\int_{-1}^2 (4y+8) - 4y^2 dy$$



8c. The volume of the solid of revolution obtained by revolving the region R about the line $x=20$.



$$\pi \int_{-1}^2 (20 - 4y^2)^2 - (20 - (4y + 8))^2 dy$$

(volume of washer is $\pi R^2 - \pi r^2$;

SINCE washers are stacked vertically, EVERYTHING is IN TERMS OF y :
outer radius length is the difference between these two x -coordinates: $20 - 4y^2$
inner radius length is the difference between THESE two x -coordinates: $20 - (4y + 8)$