Math 205A Quiz 03 page 1 September 26, 2008 NAME Suggested Solutions

1. Suppose $x_3 \begin{bmatrix} -8 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$, where x_3 and x_5 are free, is the solution $\mathbf{v}_h$ of the homogeneous matrix equation $B\mathbf{x} = \mathbf{0}$ for some matrix B. Also, let $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ be the column vectors of B.

1A. You can *not* tell from the above info how many rows B has. But how many *columns* must B have, and how do you know?

The six rows in the vectors above tell us the variables are $x_1, x_2, \ldots, x_6$, one for each column of B. $\therefore$ B has 6 columns.

(for reference in later problems, note that $\vec{v}_h$ can be written as follows $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -8x_3 + 4x_5 \\ 0x_3 + 0x_5 \\ x_3 \\ -3x_5 \\ x_5 \\ 0x_3 + 0x_5 \end{bmatrix} = \begin{bmatrix} -8x_3 + 4x_5 \\ 0 \\ x_3 \\ -3x_5 \\ x_5 \\ 0 \end{bmatrix}$

1B. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ of column vectors of B linearly independent? Explain in terms of the definition of LI (that is, consider the solutions of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots = \mathbf{0}$).

No. In order to be a L.I. set, the only solution to $x_1\vec{v}_1 + \cdots + x_6\vec{v}_6 = \vec{0}$ must be the trivial sol'n, that is, $x_1 = \cdots = x_6 = 0$. But since x_3 & x_5 are free, this eq'n has nontrivial sol'n's.

1C. Use the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots = \mathbf{0}$ to express $\mathbf{v}_1$ as a specific linear combination of the other column vectors, or explain why this is impossible.

We need specific values of $x_1, \ldots, x_6$ which satisfy this equation and where $x_1 \neq 0$. There are ∞-many ways to do this. FOR EXAMPLE, let $x_3 = x_5 = 1$. Then we also have $x_1 = -8\cdot 1 + 4\cdot 1 = \underline{-4}$, $x_2 = 0$, $x_4 = -3\cdot 1$, $x_6 = 0$. Thus, $-4\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3 - 3\vec{v}_4 + 1\vec{v}_5 + 0\vec{v}_6 = \vec{0}$ and we can "solve" for $\vec{v}_1$ to get $\vec{v}_1 = \frac{1}{4}\vec{v}_3 - \frac{3}{4}\vec{v}_4 + \frac{1}{4}\vec{v}_5$

1D. Express $\mathbf{v}_2$ as a linear combination of the other column vectors, or explain why this is impossible.

Impossible: if $\vec{v}_2 = \alpha_1\vec{v}_1 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4 + \alpha_5\vec{v}_5 + \alpha_6\vec{v}_6$ then $\vec{0} = \alpha_1\vec{v}_1 + (-1)\vec{v}_2 + \alpha_3\vec{v}_3 + \cdots + \alpha_6\vec{v}_6$, a solution of $B\vec{x} = \vec{0}$ in which $x_2 \neq 0$. But $\vec{v}_h$ represents all sol'n's of $B\vec{x} = \vec{0}$, and $\vec{v}_h$ has a zero in the 2nd row, ie x_2 is always 0.

note: although not a great answer, I accepted this: "in $x_1\vec{v}_1 + \cdots + x_6\vec{v}_6 = \vec{0}$, the coefficient of $\vec{v}_2$ is 0, so we cannot solve for $\vec{v}_2$..." I did NOT want to see any nonsence about "$\vec{v}_2 = \frac{1}{0}(\cdots)$"

1E. Let $\mathbf{b} = 7\mathbf{v}_1 + 6\mathbf{v}_2 - 12\mathbf{v}_4$ in the following two questions:

1E (i). Express $\mathbf{b}$ as a linear combination of the column vectors of B *without* using $\mathbf{v}_4$ (by replacing $\mathbf{v}_4$ with a LC of the other column vectors).

we first need to express $-12\vec{v}_4$ in terms of the other columns. now, if $x_5 = 1$ and $x_3 = 0$ we get $x_1 = -8x_3 + 4x_5 = -8\cdot 0 + 4\cdot 1 = 4$, $x_2 = 0$, $x_3 = 0$, $x_4 = -3x_5 = -3\cdot 1 = -3$, $x_5 = 1$, $x_6 = 0$

so $4\vec{v}_1 - 3\vec{v}_4 + \vec{v}_5 = \vec{0}$; $-3v_4 = -4\vec{v}_1 - \vec{v}_5 \Rightarrow -12\vec{v}_4 = -16v_1 - 4\vec{v}_5$

$\to$ so $\vec{b} = 7\vec{v}_1 + 6\vec{v}_2 - 12\vec{v}_4 = 7\vec{v}_1 + 6\vec{v}_2 + (-16\vec{v}_1 - 4\vec{v}_5) = -9\vec{v}_1 + 6\vec{v}_2 - 4\vec{v}_5$

(there are ∞-many answers, depending on how you choose x_3 & x_5)

1E (ii). Can you express $\mathbf{b}$ as a linear combination of the column vectors of B *without* using $\mathbf{v}_2$? Explain your answer.

NO, since $\vec{v}_2$ is not a L.C. of the other columns. (good enough)

BETTER: if $\vec{b}$ = L.C. of other columns, then $\vec{0} = \vec{b} - \vec{b}$ = (L.C. other columns) $- (7\vec{v}_1 + 6\vec{v}_2 - 12\vec{v}_4)$; the last expression gives a sol'n of $B\vec{x} = \vec{0}$ in which the coefficient of $\vec{v}_2$ is -6. But x_2 is always 0.

1 Challenge Bonus Question: Let m be the number of rows of B. Suppose $B\mathbf{x} = \mathbf{c}$ does *not* have a solution for every $\mathbf{c}$ in $\mathbf{R}^m$. What is the RREF of B, where m is as small as possible?

we have $\begin{cases} x_1 + 8x_3 - 4x_5 = 0 \\ x_2 = 0 \\ x_4 + 3x_5 = 0 \\ x_6 = 0 \end{cases}$ from $\vec{v}_h$; ie RREF(B) is

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 8 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where we add a row of 0's to ensure $B\vec{x} = \vec{c}$ might be inconsistent.