Math 205B Quiz 02 page 1 09/25/2009 Name suggested solutions

1. Let $\mathbf{b}$ and $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ be vectors in $\mathbf{R}^m$. Complete the following sentence so that it gives the definition of *linear combination*: "We say $\mathbf{b}$ is a linear combination of the the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ if and only if..."

... there exist scalars $\alpha_1, \cdots, \alpha_n$ in $\mathbb{R}$ such that $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{b}$.

2. Let $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$. $\mathbf{a}_4 = \begin{bmatrix} 10 \\ 4 \\ -4 \end{bmatrix}$. Also, let $\mathbf{b} = \begin{bmatrix} 16 \\ 8 \\ 26 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 26 \\ 8 \\ 16 \end{bmatrix}$.

2A. Is $\mathbf{b}$ in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$? Explain your answer. Show any matrices and corresponding rref's you use.

The augmented matrix corresponding to this question is

$$\left[\begin{array}{cccc|c} 2 & 6 & 1 & 10 & 16 \\ 1 & 3 & 0 & 4 & 8 \\ -3 & -9 & 4 & -4 & 26 \end{array}\right]. \text{ Its rref is } \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Since this matrix tells us the system of equations represented by the first matrix is inconsistent (since the last row of the rref matrix represents the equation $0x_1 + \cdots + 0x_4 = 1$) $\vec{b}$ is not in the span of $\{\vec{a}_1, \cdots, \vec{a}_4\}$

2B. Let A be the matrix whose columns are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$. Express all solutions of $A\mathbf{x} = \mathbf{c}$ in parametric vector form, that is, as $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{p}$ is a particluar solution of $A\mathbf{x} = \mathbf{c}$ and $\mathbf{v}_h$ is all solutions of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$. Show any relevant matrices used in your work. (circle $\vec{p}$ and $\vec{v}_h$)

The augmented matrix now is

$$\left[\begin{array}{cccc|c} 2 & 6 & 1 & 10 & 26 \\ 1 & 3 & 0 & 4 & 8 \\ -3 & -9 & 4 & -4 & 16 \end{array}\right]; \text{ its rref is } \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 8 \\ 0 & 0 & 1 & 2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

there are infinitely many solutions, given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 - 3x_2 - 4x_4 \\ x_2 \\ 10 - 2x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ 0 \\ 10 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_2\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\vec{V}_h},$$

where x_2 & x_4 are free.

2C. Use your work in (2B) to find two nontrivial solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ of $A\mathbf{x} = \mathbf{0}$. CIRCLE your answers.

Remember that $\vec{v}_h$ gives all solutions of $A\vec{x} = \vec{0}$ and to get specific solutions we just need to choose values for x_2 & x_4. For example, take $x_2 = 2$ and $x_4 = 1$;

$$\vec{V}_h \text{ becomes } 2\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

(note the $\vec{p}$ of 2B is NOT involved since we are NOT looking for solutions of $A\vec{x} = \vec{c}$!!!)

2D. Now let T be the matrix whose columns are $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_4$ (so T looks like A if you take out A's second column). What is v_h now? That is, what are the solutions of $T\mathbf{x} = \mathbf{0}$?

(we'll let the corresponding variables be x_1 x_3 & x_4)

$$\text{now, } \left[T \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 10 & 0 \\ 1 & 0 & 4 & 0 \\ -3 & 4 & -4 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_4 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x_1 = -4x_4 \\ x_2 = -2x_4 \\ x_4 \text{ is free} \end{array} \Rightarrow \vec{v}_h = x_4\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

(here $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$)