Math 106: Review for Exam I - SOLUTIONS

1. **Find the following.** [Substitution tip: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.]

   (a) Let $u = \sqrt{x}$, so $du = \dfrac{dx}{2\sqrt{x}}$ and $2\,du = \dfrac{dx}{\sqrt{x}}$.

$$\begin{aligned}\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int_{x=1}^{x=4} e^u \cdot 2\,du \\ &= 2e^u\Big|_{x=1}^{x=4} \\ &= 2e^{\sqrt{x}}\Big|_1^4 \\ &= 2e^2 - 2e \ (\approx 9.342)\end{aligned}$$

If you prefer to switch the limits, use $u = 1$ to $u = 2$.

   (b) Let $u = \cos(5x)$, so $du = -5\sin(5x)$ and $-\dfrac{du}{5} = \sin(5x)$.

   This time, we'll change the limits:

   $x = \pi \Rightarrow u = \cos(5\cdot\pi) = -1$ and $x = 2\pi \Rightarrow u = \cos(5\cdot 2\pi) = 1$

$$\begin{aligned}\int_\pi^{2\pi} \cos^7(5x)\sin(5x)\,dx &= \int_{-1}^1 u^7 \cdot \frac{-du}{5} \\ &= -\frac{1}{5}\int_{-1}^1 u^7\ du \\ &= -\frac{1}{5}\frac{u^8}{8}\Big|_{-1}^1 \\ &= -\frac{1}{40}\left[\frac{1^8}{8} - \frac{(-1)^8}{8}\right] \\ &= 0\end{aligned}$$

   (c) Use $u = x^3$, so $du = 3x^2\ dx$ and $\dfrac{du}{3} = x^2\ dx$.

$$\begin{aligned}\int \frac{7x^2}{1+x^6}\,dx &= 7\int \frac{\frac{du}{3}}{1+u^2} \\ &= \frac{7}{3}\arctan u + C \\ &= \frac{7}{3}\arctan(x^3) + C\end{aligned}$$

   (d) Use #42 from the table of integrals with $n = 3$ and $a = 5$.

$$\begin{aligned}\int \cos^3(5x)\,dx &= \frac{\cos^2(5x)\sin(5x)}{3\cdot 5} + \frac{2}{3}\int \cos(5x)\ dx \\ &= \frac{\cos^2(5x)\sin(5x)}{15} + \frac{2}{3}\int \cos(u)\frac{du}{5} \\ &= \frac{\cos^2(5x)\sin(5x)}{15} + \frac{2}{3}\cdot\frac{1}{5}\sin(u) + C \\ &= \frac{\cos^2(5x)\sin(5x)}{15} + \frac{2}{15}\sin(5x) + C\end{aligned}$$

Subsitute $u = 5x$, so $du = 5\ dx$.

(e) Use $u = 10 - x$, so $du = -dx$ and $dx = -du$.
We'll change the limits:
$x = 6 \Rightarrow u = 10 - x = 4$ and $x = 10 \Rightarrow u = 10 - 10 = 0$

$$\int_6^{10} x\sqrt{10-x}\,dx = \int_4^0 (10-u)\sqrt{u}(-du)$$

Since $u = 10 - x$, we know $x = 10 - u$.

$$= \int_4^0 (u-10)\sqrt{u}\,du$$
$$= \int_4^0 (u^{3/2} - 10u^{1/2})du$$
$$= \left[\frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2}\right]_4^0$$
$$= (0-0) - \left(\frac{2}{5}4^{5/2} - \frac{20}{3}4^{3/2}\right)$$
$$= \frac{608}{15} = 40.5\overline{3}$$

NOTE: The 9:30 and 2:40 sections should omit the midpoint (M_n), trapezoidal (T_n), and Simpson's (S_n) parts of problems on this and the following pages.

2. If $f(x)$ is decreasing and concave up, put the following quantities in ascending order.

L_{100}, R_{100}, T_{100}, M_{100}, $\int_a^b f(x)\,dx$

$R_{100} < M_{100} < \int_a^b f(x)\,dx < T_{100} < L_{100}$

What can you say with certainty about where S_{200} would fit into your list above?

It would be somewhere between M_{100} and T_{100} but we don't know how it compares to $\int_a^b f(x)\,dx$.

3. **Suppose $f(t)$ is the rate of change (in animals per month) of a population $P(t)$.**

(a) **What does $\int_4^{12} f(t)\,dt$ represent in this problem?**

It represents the total (or net) change in the number of animals during the time period $[4, 12]$.

(b) **Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to $\int_4^{12} f(t)\,dt$ given the data in the table below.**

t	4	6	8	10	12
$f(t)$	15	11	8	4	3

$L_4 = (15 + 11 + 8 + 4)(2) = 76$ $\quad R_4 = (11 + 8 + 4 + 3)(2) = 52$ $\quad T_4 = \frac{L_4 + R_4}{2} = 64$

We cannot compute M_4 because it requires the values of f at $x = 5$, 7, 9, and 11. Instead, we do M_2.

$M_2 = (11 + 4)(4) = 60$ Now, to find S_4, we need $T_2 = \frac{L_2 + R_2}{2} = \frac{(15+8)(4) + (8+3)(4)}{2} = 68$.

$S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.\overline{6}$

4. **Find bounds for each of the following errors if $I = \int_2^7 \ln x\,dx$.**

(a) $|I - L_{100}| \le \dfrac{K_1(b-a)^2}{2n} = \dfrac{\frac{1}{2}(7-2)^2}{2(100)} = \dfrac{1}{16}$

$K_1 =$ max of $|f'(x)|$ on $[2,7] =$ max of $\dfrac{1}{x}$ on $[2,7] = \dfrac{1}{2}$ (occurs at $x = 2$)

(b) $|I - T_{100}| \le \dfrac{K_2(b-a)^3}{12n^2} = \dfrac{\frac{1}{4}(7-2)^3}{12(100)^2} = \dfrac{1}{3840}$

$K_2 =$ max of $|f''(x)|$ on $[2,7] =$ max of $\dfrac{1}{x^2}$ on $[2,7] = \dfrac{1}{4}$ (occurs at $x = 2$)

(c) $|I - M_{100}| \le \dfrac{K_2(b-a)^3}{24n^2} = \dfrac{\frac{1}{4}(7-2)^3}{24(100)^2} = \dfrac{1}{7680}$

$K_2 =$ same as in previous part

5. **If $I = \int_2^7 \ln x \, dx$, how many subdivisions are required to obtain a left-hand sum approximation with error of at most 0.001?**

From part (a) above, we know that $|I - L_n| \le \dfrac{K_1(b-a)^2}{2n} = \dfrac{\frac{1}{2}(7-2)^2}{2n} = \dfrac{25}{4n}$.

Thus, we want $\dfrac{25}{4n} \le 0.001$, which is equivalent to $\dfrac{25}{4} \le 0.001n$, which is equivalent to $\dfrac{25000}{4} \le n$.

Since $\dfrac{25000}{4} = 6250$, we must use at least 6250 subdivisions.

6. **Use Euler's method with three steps on the differential equation $\dfrac{dy}{dt} = y - t$ to estimate $y(2.5)$ if $y(1) = 0$.**

t	y	$\frac{dy}{dt} \cdot \Delta t = \Delta y$
1	0	(-1)(0.5) = -0.5
1.5	-0.5	(-2)(0.5) = -1
2	-1.5	(-3.5)(0.5) = -1.75
2.5	-3.25	

So, $y(2.5) \approx -3.25$ (or -13/4).

7. **Solve the differential equation $dy/dx = 2xy + 6x$ if the solution passes through $(0, 5)$.**

$$\frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = 2x(y+3)$$

$$\frac{dy}{y+3} = 2x\ dx \qquad \text{Separate the variables.}$$

$$\int \frac{dy}{y+3} = \int 2x\ dx$$

$$\ln|y+3| = x^2 + C$$

$$|y+3| = e^{x^2+C} \qquad \text{Exponentiate each side to remove the ln.}$$

$$y + 3 = \pm e^C e^{x^2} \qquad |w| = z \text{ means } w = \pm z.$$

$$y = -3 + Ae^{x^2} \qquad \text{Replace } \pm e^C \text{ with } A.$$

Now we use the initial condition $y(0) = 5$ to find the value of A.

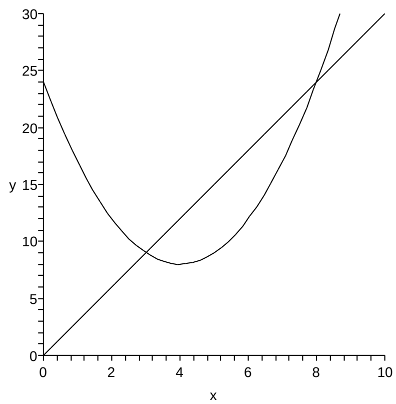
We have $5 = -3 + Ae^0 \Rightarrow A = 8$, so the solution is $y = -3 + 8e^{x^2}$.

8. **Write integrals equal to**

(a) **the arc length of $y = x^2$ on the interval $[1, 5]$**

arc length of $y = f(x)$ on $[a, b] = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_1^5 \sqrt{1 + (2x)^2}\, dx$ (≈ 24.395)

(b) **the area bounded by $y = x^2 - 8x + 24$ and $y = 3x$**



First, find where the curves intersect.

$$\begin{aligned} x^2 - 8x + 24 &= 3x \\ x^2 - 11x + 24 &= 0 \\ (x-3)(x-8) &= 0 \\ &\Rightarrow x = 3, x = 8 \end{aligned}$$

Between $x = 3$ and $x = 8$, $y = 3x$ is above $y = x^2 - 8x + 24$. (Plug in $x = 5$ or graph to check.)

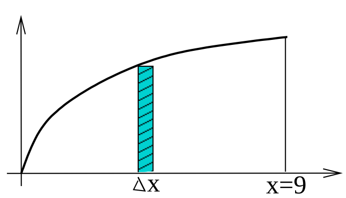
So, the area between them is

$$\int_3^8 [3x - (x^2 - 8x + 24)]\, dx.$$

[This equals 125/6.]

9. **Consider the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$. Write an integral equal to the volume generated if this region is rotated about**

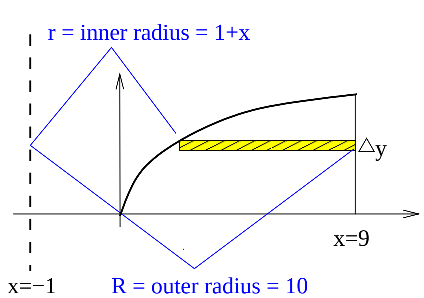
(a) **the x-axis**



$$\begin{aligned} \text{volume of slice } &\approx \pi r^2 \Delta x \\ &= \pi y^2 \Delta x \\ &= \pi(\sqrt{x})^2 \Delta x \\ &= \pi x \Delta x \end{aligned}$$

$$\text{total volume} = \pi \int_0^9 x\, dx$$

(b) **the line $x = -1$**



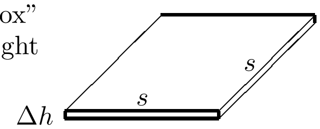
$$\begin{aligned} \text{volume of slice } &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\ &= \pi(10^2)\Delta y - \pi(1+x)^2 \Delta y \\ &= \pi[100 - (1+y^2)^2]\Delta y \end{aligned}$$

$$\text{total volume} = \pi \int_0^3 [100 - (1+y^2)^2]\, dy$$

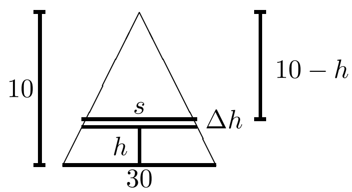
10. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) **the volume of the pyramid**

We slice horizontally, so each slice is a "box" with a square top and bottom and a height (thickness) of Δh, as shown to the right.



The picture shown below is a vertical cross-section through the center of the pyramid.



Similar triangles: $\dfrac{10}{30} = \dfrac{10-h}{s} \Rightarrow s = 3(10-h)$.

volume of slice $\approx s^2\Delta h \approx [3(10-h)]^2\Delta h$

total volume $= \displaystyle\int_0^{10} [3(10-h)]^2\, dh$

(b) **the work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water (62.4 pounds per cubic foot)**

We use the same sketch as in the previous part.

$$\begin{aligned}
\text{volume of slice } &\approx s^2\Delta h \approx [3(10-h)]^2\Delta h && \text{From above.}\\
\text{weight of slice } &\approx 62.4[3(10-h)]^2\Delta h && \text{Weight=(density)(volume).}\\
\text{work to lift slice } &\approx 62.4[3(10-h)]^2\Delta h(15-h) && \text{Work=(force)(distance); here, force=weight.}\\
\text{total volume } &= 62.4\int_0^8 [3(10-h)]^2(15-h)\, dh
\end{aligned}$$