Math 205A Quiz 01, page 1 September 14, 2007 NAME Suggested Solutions

1. Suppose $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 3 \\ -13 \\ -8 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -6 \\ 9 \\ -39 \\ -24 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -3 \\ 3 \\ 2 \end{bmatrix}$.

1A. What is the (initial, non-row reduced) augmented matrix which corresponds to the system of equations you need to solve in order to find out if the vector $\mathbf{b} = \begin{bmatrix} 15 \\ -31 \\ 77 \\ 50 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$?

$$\left[\begin{array}{cccc|c} 1 & -2 & -6 & 1 & 15 \\ -2 & 3 & 9 & -3 & -31 \\ 5 & -13 & -39 & 3 & 77 \\ 4 & -8 & -24 & 2 & 50 \end{array}\right]$$

NOTE! We will always write the "|" in an augmented matrix (the book does not)

1B. An augmented matrix which is row equivalent to the correct answer to (1A) is given below; you do *not* need to verify this. Use it to determine if $\mathbf{b}$ is indeed in the span of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$. If $\mathbf{b}$ is not in the span, explain why not. If it is in the span, find all the linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ which add up to $\mathbf{b}$ (that is, find the solution of the system of equations refered to in (1A); if there are any solutions write them in terms of the free variables (if there are any)).

Row equivalent matrix: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

This augmented matrix shows no inconsistency, so that $\vec{b}$ is in the span of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$; the solution of the system refered to is

$$\begin{cases} x_1 = 2 \\ x_2 = -4 - 3x_3 \\ x_3 \text{ is free} \\ x_4 = 5 \end{cases}$$

The presence of the free variable means there are ∞-many ways to express $\vec{b}$ as a L.C. of $\vec{u}_1, \ldots, \vec{u}_4$

1C. If the vector **b** had been $\begin{bmatrix} 15 \\ -31 \\ 77 \\ 57 \end{bmatrix}$ instead, the corresponding augmented matrix would've been different as shown below. Would this new **b** be in the span of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$? Explain why or why not.

New row equivalent augmented matrix: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$.

This matrix has the inconsistency $0=7$, so there are no solutions to

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4 = \begin{bmatrix} 15 \\ -31 \\ 77 \\ 57 \end{bmatrix}; \text{ i.e. THIS } \vec{b}$$

is NOT in the span of $\vec{u}_1, \dots, \vec{u}_4$.

1D. Is there a vector **b** in the span of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ which can only be written as a linear combination of these four vectors in *exactly one way*? Or if there's at least one way, will there be infinitely many? Explain. (Hint: no matter what **b** you attempt to write as a LC of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$, will there be any difference in the first four columns of the reduced echelon form of the resulting augmented matrix? Look at the first four columns of those given in (1B) and (1C))

starting with ANY arbitrary $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, solving $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4 = \vec{b}$

will always result in an augmented matrix of the form $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1' \\ 0 & 1 & 3 & 0 & b_2' \\ 0 & 0 & 0 & 1 & b_3' \\ 0 & 0 & 0 & 0 & b_4' \end{array}\right]$.

(where the b_i''s will be various combinations of the b_i's).

EITHER b_4' will be zero, or it won't: NOW, consider these two possibilities:

① If $b_4' \neq 0$ then the corresponding inconsistency means $\vec{b} \notin$ span.

② If b_4' IS 0, there will not be an inconsistency, so $\vec{b} \in$ span AND FURTHERMORE, the presence of the free variable x_3 means there are ∞-many ways to express $\vec{b}$ as a L.C. of the $\vec{u}_i$'s; FOR NO $\vec{b}$ is there EXACTLY ONE WAY!