NAME____

I____II____III____IV____V____VI____VII____VIII____ IX____X____TOTAL _________

I	II	III	IV	V	VI	VII	VIII	IX	X	TOTAL
(15)	(10)	(10)	(15)	(5)	(5)	(10)	(5)	(15)	(10)	(100)

April 17, 2009

Mathematics 309
Abstract Algebra I
Final Examination

Mr. Haines

(15) I. Define any three of these terms. Use a complete, mathematically correct sentence for each definition.

binary operation on a set S
integral domain
unit of a ring
coset of a subgroup H of a group G

A.

B.

C.

(10) II. Give examples of the following if possible. If not possible, say why.

A. An integral domain with an even number of elements.

B. An infinite cyclic subgroup of $\langle C^{*}, \cdot \rangle$, the non-zero complex numbers under multiplication.

C. A finite non-trivial cyclic subgroup of $\langle C^{*}, \cdot \rangle$, the non-zero complex numbers under multiplication.

D. A commutative ring without unity.

E. Three non-isomorphic groups, each with 27 elements.

F. A field that is not an integral domain.

G. A polynomial of degree 5 in $\mathbb{Z}[x]$ that is irreducible in $\mathbb{Z}[x]$.

H. A non-trivial homomorphism from Z_9 to Z_7.

I. A homomorphism from Z_4 to Z_{64} whose kernel is $\{0\}$.

(10) III. Complete these two tables to give operations for two non-isomorphic groups of order 4. Assume that e is the identity for both groups.

A.

$*$	e	a	b	c
e				
a				
b				
c				

B.

$*'$	e	a	b	c
e				
a				
b				
c				

(15) IV. Let p be a prime and H be the set of all zeroes in C to the polynomial $z^p - 1$.

A. Prove that H is a subgroup of the group of non-zero complex numbers under multiplication.

B. Prove that H is abelian.

C. Classify H according to the Fundamental Theorem of Finitely Generated Abelian Groups.

(5) V. If H is a subgroup of the group G and $a \in G$, prove that H and aH have the same number of elements by constructing a function from H to aH and proving it is one-to-one and onto.

(5) VI. Draw the lattice diagram of subgroups of Z_{54}, the integers modulo 54, where the operation is addition modulo 54.

(10) VII.

A. Prove that if $\phi : Z_n \rightarrow Z_n$ is a group homomorphism, then once you know the value of $\phi(1)$, then you know the value of ϕ for all elements of its domain.

B. Using Part A, compute the number of non-trivial homomorphisms from Z_{11} to Z_{11}.

(5) VIII. Factor $x^5 + x^2 + x + 1$ into irreducible factors over $\mathbb{Z}_2[x]$

(15) IX. Fill in the blanks:

A. A generator for $Z_8 \times Z_7$ different from (1, 1) is ________________________ .

B. The order of (1, 2, 3)(2, 3, 4)(1, 2) in S_5 is _________________________ .

C. The number of zeroes of $x^3 + x + 1$ in $\mathbb{Z}_3$ is ___________________.

D. The number of left cosets of $\langle 7 \rangle$ in Z_{20} is ____________________.

E. The number of elements in the group S_6 is ______________________ .

F. The order of the group D_5 of symmetries of the regular pentagon is _____________ .

G. Express $(1, 5, 7, 3, 4) \in S_7$ as a product of transpositions __________________ .

H. The order of $3 + \langle 4 \rangle$ in the group $Z_{12} / \langle 4 \rangle$ is __________________.

I. The index of A_7 in S_7 is ____________________________________ .

J. The number of right cosets of $\langle 5 \rangle$ in Z_{10} is ____________________.

K. The order of the group $(Z_4 \times Z_8)/ \langle (2, 2) \rangle$ is ______________________ .

L. The units in the ring $\langle Z_{18}, +_{18}, \cdot_{18} \rangle$ are ______________________ .

M. Cayley's Theorem: Every group is isomorphic to a group of _____________________ .

(10) X. TRUE OR FALSE? (The number of incorrect responses will be subtracted from the number of correct ones. Random guessing earns you no points at all.)

______ 1. The real numbers are closed under division.

______ 2. The empty set is an example of a ring.

______ 3. Every cyclic group has more than one generator.

______ 4. The real numbers form an abelian group under addition.

______ 5. The real numbers form a group under multiplication.

______ 6. $Z_5 \times Z_{28}$ is a cyclic group.

______ 7. The number of elements in any subgroup of a finite group G divides the number of elements in G .

______ 8. Every permutation can be expressed as a product of cycles.

______ 9. S_6 has no cyclic subgroups.

______ 10. The composition of two permutations of a non-empty set A is always a permutation of A .

______ 11. Every left coset of a subgroup of a group G is also a subgroup of G .

______ 12. Every abelian group of order 8 contains a cyclic subgroup of order 8 .

______ 13. Every finite group of prime order is cyclic

______ 14. If $H \leq S_5$ and H has 30 elements,then S_5 / H is abelian.

______ 15. The function $\phi : Z_6 \rightarrow Z_{12}$ defined by $\phi(n) = n$ for all $n \in Z_6$ is a homomorphism.

______ 16. $x^7 + x^5 + x^4 + x^3 + x + 1$ is irreducible over $\mathbb{Z}_3[x]$.

______ 17. Every coset of a group G is also a subgroup of G .

______ 18. For any two groups G and G' there is an isomorphism from G to G' .

______ 19. Every field is an integral domain.

______ 20. Every finite group is cyclic.