Useful Information for problem 3

$$[A|I_5] = \left[\begin{array}{rrrrrr|rrrrr} 1 & 1 & 2 & 3 & 10 & 18 & 1 & 0 & 0 & 0 & 0 \\ 5 & 12 & 10 & 15 & 50 & 146 & 0 & 1 & 0 & 0 & 0 \\ 3 & 6 & 6 & 9 & 30 & 78 & 0 & 0 & 1 & 0 & 0 \\ 5 & 10 & 9 & 13 & 46 & 124 & 0 & 0 & 0 & 1 & 0 \\ 5 & 11 & 8 & 11 & 42 & 126 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrrrrr|rrrrr} 1 & 0 & 0 & -1 & 2 & -2 & 0 & 0 & -19/3 & 6 & -2 \\ 0 & 1 & 0 & 0 & 0 & 8 & 0 & 0 & 5/3 & -2 & 1 \\ 0 & 0 & 1 & 2 & 4 & 6 & 0 & 0 & 5/3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4/3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & 4 & -2 \end{array}\right]$$

$$A^T = \left[\begin{array}{rrrrr} 1 & 5 & 3 & 5 & 5 \\ 1 & 12 & 6 & 10 & 11 \\ 2 & 10 & 6 & 9 & 8 \\ 3 & 15 & 9 & 13 & 11 \\ 10 & 50 & 30 & 46 & 42 \\ 18 & 146 & 78 & 124 & 126 \end{array}\right] \sim \left[\begin{array}{rrrrr} 1 & 0 & 6/7 & 0 & -15/7 \\ 0 & 1 & 3/7 & 0 & -4/7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1a. The points $(1, 11)$, $(2, 10)$, $(4, 14)$ and $(5, 19)$ do all belong to a parabola of the form $\beta_2 x^2 + \beta_1 x + \beta_0$. Find β_2, β_1 and β_0 by setting up and solving an appropriate system of linear equations.

1b. The data points gathered in a lab experiment were *supposed* to lie on a parabola of the form $\beta_2 x^2 + \beta_1 x + \beta_0$. The points were $(1, 12)$, $(2, 9)$, $(4, 19)$ and $(5, 22)$. Show why there can be no such parabola.

1c. Find the best fit (least-squares) parabola of the form $\beta_2 x^2 + \beta_1 x + \beta_0$ for the data points in (1b). Show all your work.

(Problem (1) continued)

1d. What is the vector of predicted values given by the answer to (1c)?

1e. What is the resulting residual vector? And what is the sum-of-squares of the residuals?

1f. Why is the answer to (1c) a "better fit" for the data points in (1b) than the parabola obtained in (1a) is for those same data points in (1b)?

2. Suppose that M is a 3 by 3 matrix and $\begin{bmatrix} -2 \\ 5 \\ 6 \end{bmatrix}$ is in its null space.

Suppose also that both $\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ satisfy $M\mathbf{x} = 4\mathbf{x}$.

2a. What is the dimension of the eigenspace corresponding to $\lambda = 4$? Explain.

2b. What is the characteristic polynomial of M?

2c. Find M (Hint: first find "P and D" as in the diagonalization theorem. Show P and D as part of your work).

3. Let $A = \begin{bmatrix} 1 & 1 & 2 & 3 & 10 & 18 \\ 5 & 12 & 10 & 15 & 50 & 146 \\ 3 & 6 & 6 & 9 & 30 & 78 \\ 5 & 10 & 9 & 13 & 46 & 124 \\ 5 & 11 & 8 & 11 & 42 & 126 \end{bmatrix}$; see the useful information on page 0.

Label the six columns of A as $\mathbf{a}_1, \ldots, \mathbf{a}_6$. In your answers, wherever possible, use the "$\mathbf{a}_1, \ldots, \mathbf{a}_6$" notation *instead of explicitly writing out the column vectors.*

3a. Find a basis $\mathcal{B}$ for $\text{Col}(A)$.

3b. Write $\mathbf{a}_6$ as a linear combination of the basis vectors in $\mathcal{B}$.

3c. Let R be the reduced row echelon form (RREF) of A. Is the first column of R in $\text{Col}(A)$? Explain how you know. (*Hint:* Try to express that first column as a linear combination of the members of $\mathcal{B}$). Does $\text{Col}(A) = \text{Col}(R)$? Explain.

3d. Find all solutions of $A\mathbf{x} = \mathbf{a}_6$ expressed in the form $\mathbf{p}+\mathbf{v}_h$ where $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{a}_6$ and $\mathbf{v}_h$ is all solutions of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$. (Note: the information on page 0 should be sufficient; you shouldn't need to do any additional row reductions)

Problem 3 continued.

3e. Find a basis $\mathcal{D}$ of $(\mathrm{Col}(A))^{\perp}$.

3f. Verify that the first vector in $\mathcal{D}$ is perpendicular to the first vector in $\mathcal{B}$ from (3a).

3g. Let the rows of A be labeled $\mathbf{z}_1, \ldots, \mathbf{z}_5$. Show how to write $\mathbf{z}_3$ as a linear combination of the first two rows of A. *Hint:* Remember, $\mathrm{Row}(A) = \mathrm{Col}(A^T)$ (except for horizontal *vs* vertical orientation); use the RREF of A^T to find a relationship between the first three columns of A^T)

3h. Can the first three rows of A be a basis for $\mathrm{Row}(A)$? Explain.

3i. We know that $\mathrm{Row}(A) = \mathrm{Row}(R)$. Indeed, show how to write $\mathbf{z}_1$ as a linear combination of the rows of R.

5. Let $\mathbb{F}$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ as discussed in class.
Let $H = \{f \in \mathbb{F} \mid \text{ if } a < b \text{ then } f(a) \leq f(b)\}$.

5a. Which of the following functions is in H? (Circle them)

$\mathbf{u}_1 = e^x$ $\qquad$ $\mathbf{u}_2 = x^2 + 1$ $\qquad$ $\mathbf{u}_3 = \sqrt[3]{x}$

5b. Is the zero-vector of $\mathbb{F}$ in H? Explain.

5c. Is H closed under vector addition? Show why or give a counterexample.

5d. Is H closed under scalar multiplication? Show why or give a counterexample.

6. Suppose that V and W are vector spaces and $T : V \to W$. What does it mean to say T is a *linear transformation*?