This page contains useful information for problem 1.

The Reduced Row Echelon Form (rref) of

$$\left[\begin{array}{rrrrr|rrrr|rr} 2 & -4 & -4 & -2 & 6 & 2 & 0 & 6 & 4 & 5 & 6 \\ -1 & 2 & 2 & 1 & -3 & -1 & 0 & -3 & -2 & 3 & -3 \\ 4 & -3 & 7 & 1 & 1 & 1 & -1 & 5 & 7 & 5 & 4 \\ 1 & 1 & 7 & 2 & 4 & 3 & 7 & -5 & 9 & 3 & 21 \end{array}\right]$$

is

$$\left[\begin{array}{rrrrr|rrrr|rr} 1 & 0 & 4 & 1 & 0 & 1/2 & 1 & -1/2 & 3 & 0 & 4 \\ 0 & 1 & 3 & 1 & 0 & 1/2 & 2 & -5/2 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1/2 & 1 & -1/2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The Reduced Row Echelon Form (rref) of

$$\left[\begin{array}{rrrr|rrrrr|rr} 2 & 0 & 6 & 4 & 2 & -4 & -4 & -2 & 6 & 5 & 6 \\ -1 & 0 & -3 & -2 & -1 & 2 & 2 & 1 & -3 & 3 & -3 \\ 1 & -1 & 5 & 7 & 4 & -3 & 7 & 1 & 1 & 5 & 4 \\ 3 & 7 & -5 & 9 & 1 & 1 & 7 & 2 & 4 & 3 & 21 \end{array}\right]$$

is

$$\left[\begin{array}{rrrr|rrrrr|rr} 1 & 0 & 3 & 0 & 0 & -2 & -6 & -2 & 4 & 0 & 2 \\ 0 & 1 & -2 & 0 & -1/2 & 1 & 1 & 1/2 & -1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1/2 & 0 & 2 & 1/2 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The RREF of the transpose of P in problem 1 is $\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1. Use the sheet of Row-Reduced Matrices (page "A") to help answer these questions. You shouldn't need a calculator in *any* part of problem 1.

$$\text{Let } P = \begin{bmatrix} 2 & -4 & -4 & -2 & 6 \\ -1 & 2 & 2 & 1 & -3 \\ 4 & -3 & 7 & 1 & 1 \\ 1 & 1 & 7 & 2 & 4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 2 & 0 & 6 & 4 \\ -1 & 0 & -3 & -2 \\ 1 & -1 & 5 & 7 \\ 3 & 7 & -5 & 9 \end{bmatrix}.$$

Label the column vectors of P as $\mathbf{p}_1, \ldots, \mathbf{p}_5$, and those of Q as $\mathbf{q}_1, \ldots, \mathbf{q}_4$.

1A. Find a basis $\mathcal{B}$ for Col(P). Write your answer using the symbols $\mathbf{p}_1, \ldots, \mathbf{p}_5$ (don't write out the actual column vectors).

1B. Find a basis $\mathcal{C}$ for Col(Q), using the symbols $\mathbf{q}_1, \ldots, \mathbf{q}_4$.

1C. Let $\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 4 \\ 21 \end{bmatrix}$. Use the RREF sheet to completely solve $P\mathbf{x} = \mathbf{v}$; write your answer in the "$\mathbf{x} = \mathbf{p} + \mathbf{v}_h$" notation.

1D. Find both $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{C}}$; clearly identify which is which. This is *not* a change of basis problem.

(Problem 1 continues here)

1E. In terms of the symbols $\mathbf{p}_1, \ldots, \mathbf{p}_5$, and $\mathbf{q}_1, \ldots, \mathbf{q}_4$, what superaugmented matrix represents the problem of expressing each of the basis vectors in $\mathcal{B}$ as linear combinations (LC's) of those in $\mathcal{C}$?

1F. Explicitly find the rref of the matrix in 1E. (The info you need is on the RREF sheet!)

1G. Explicitly give the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

1H. Use the rref of P to find a basis (call it $\mathcal{D}$) for Row(P).

1I. Use P^T and its rref to find another basis (call this one $\mathcal{E}$) for Row(P).

(Problem 1 continues here)

1J. Let $\mathbf{r}_4$ be the fourth row vector in P. What is $[\mathbf{r}_4]_{\mathcal{D}}$? What is $[\mathbf{r}_4]_{\mathcal{E}}$? (Identify which answer is which).

1K. Find a basis for $\mathrm{Nul}(P)$.

1L. Find a basis for $\mathrm{Col}(P)^{\perp}$.

2. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 2 \\ -1 & -1 & 4 \end{bmatrix}$

2A. Find in factored form, the characteristic polynomial of A.

2B. State the eigenvalues of A and their multiplicities.

3. Let $C = \begin{bmatrix} 5 & 0 & -1 \\ 3 & 4 & -3 \\ 5 & 0 & -1 \end{bmatrix}$.

Facts: (1) C is not invertible (2) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of C. (3) C is diagonalizable.

Find P and D such that P is invertible, D is diagonal, the columns of P are eigenvectors for C, and $C = PDP^{-1}$. Show all your work, including any matrices you require, both before and after rref-ing.

4. There is no parabola of the form $\beta_2 x^2 + \beta_1 x + \beta_0$ that contains all the points $(3,3)$, $(2,7)$, $(0,1)$ and $(-1,11)$. You do *not* need to verify this.

4A. Find the best-fit parabola of this form for these four points. Identify your design matrix, parameter vector, and observation vector. Show all your work.

4B. What are the four y-coordinates on the best-fit parabola corresponding to the x coordinates of the four points $(3,3)$, $(2,7)$, $(0,1)$ and $(-1,11)$?

4C. What is the distance from the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 1 \\ 11 \end{bmatrix}$ to the column space of the matrix $\begin{bmatrix} 9 & 3 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$?

4D. How does the answer to 4C relate to the points $(3,3)$, $(2,7)$, $(0,1)$ and $(-1,11)$ and the best-fit parabola from 4A?

5. Let $\mathbb{F}$ be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let H be the subset of all members of $\mathbb{F}$ which have a non-negative y-intercept. (The y-intercept of a function f is the y-coordinate where the graph of f crosses the y-axis).

For each part of the definition of subspace, show the H satisfies that part (give a proof) or give an explicit counterexample that H does not satisfy that part.

6. Define what it means for a set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ to be *linearly independent.*

7. Define what it means for a transformation T from $\mathbb{R}^k$ to $\mathbb{R}^j$ to be a *linear* transformation.

8. Suppose P in problem 1 is the matrix of a Linear Transformation $T : \mathbb{R}^k \to \mathbb{R}^j$.

8A. What are k and j? $k =$ $j =$

8B. Is T a one-to-one linear transformation? Explain.

8C. Is T onto $\mathbb{R}^j$? Explain.