1a) The three points $(1,5)$, $(2,3)$, and $(3,-5)$ all lie on one parabola of the form $y = a + bx + cx^2$. What system of equations in the unknowns a, b and c needs to be solved in order to find a, b, and c?

1b) Write the above system in augmented matrix form, and proceed to put that matrix in *row-reduced* echelon form.

1c) What are the values of a, b, and c? So, what is the parabola?

2a) Refer to problem 1. Now *set up* the design matrix X, observation vector $\mathbf{y}$ and unknown-parameter vector β which corresponds to a best-fit parabola of the form $y = \beta_1 x + \beta_2 x^2$ (ie, a parabola with no constant term) for those three data points $(1, 5)$, $(2, 3)$, and $(3, -5)$.

2b) Find β_1 and β_2. Show all steps, including any multiplications involving matrix transposes. Note: at the end of the work, you may find it easier to use the formula for the inverse of a 2×2 matrix, rather than row reduction (to avoid working with fractions, keep the "1/det" outside; don't multiply through by it.).

2c) In terms of the sum of residuals' squares, how "far" is this best-fit parabola from the data? Show your work.

3a) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Do the columns of A form an orthogonal set? Explain.

3b) Is the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 4 \\ 5 \end{bmatrix}$ in Col(A)? Explain why not.

3c) Find the least-squares solutions of $A\mathbf{x} = \mathbf{b}$. The answer works out nicely, so be careful. Show all your steps.

3) (*continued* from the previous page)

3d) Find both the projection $\mathbf{p}$ of $\mathbf{b}$ onto Col(A) and the vector $\mathbf{z} \in$ Col(A))$^{\perp}$ such that $\mathbf{b} = \mathbf{p}+\mathbf{z}$.

3e) Explain why finding a basis for Col(A))$^{\perp}$ is the same as finding a basis for Null(Col(A^T)).
Hint: Col(A))$^{\perp}$ consists of all vectors $\mathbf{v}$ which are $^{\perp}$ to all the column vectors of A. That means $\mathbf{c} \cdot \mathbf{v}$=0 for each column $\mathbf{c}$ of A. Columns of A turn into rows of A^T. When you find the product $A^T\mathbf{v}$, you multiply the rows of A^T by the column $\mathbf{v}$ — that is, you're finding dot products of *what* with *what*?

3f) Find a basis for Col(A))$^{\perp}$.

4) Let $A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. Explain why the columns of A form an orthogonal set.

4b) Find the projection of $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}$ onto the column space of A. Use the appropriate dot product formula. Show all your work.

5) Recall $M_{2\times2}$ is the set of all 2-by-2 matrices. Suppose H is the subset of $M_{2\times2}$ consisting of all matrices of the form $\mathbf{v} = \begin{bmatrix} a & 0 \\ 3a+2b & b \end{bmatrix}$. Prove H is a sub*space* of $M_{2\times2}$ or show by counterexample that it is not.

6) Let $V = M_{2\times 2}$ and let $W = \mathbf{P}_2$. Let $\mathcal{B} = \{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ be a basis for V. Call these basis elements $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and $\mathbf{b}_4$.

Let $\mathcal{C} = \{t^2 + 1, \quad t, \quad t + 1\}$ be a basis for W; name these basis elements $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$. Let $T : V \to W$ be the linear transformation defined by $T(\mathbf{b}_1) = 2t^2 + 3t + 2$, $T(\mathbf{b}_2) = 5t$, $T(\mathbf{b}_3) = t^2 + 2t + 2$, and $T(\mathbf{b}_4) = 3t + 3$.

6a) Find $[T(\mathbf{b}_1)]_{\mathcal{C}}$.

6b) Find the matrix P of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

6c) Let $\mathbf{v} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$. Find $[\mathbf{v}]_{\mathcal{B}}$.

6d) Use the matrix P to find $[T(\mathbf{v})]_{\mathcal{C}}$.

6e) Explcitly, what polynomial is $T(\mathbf{v})$?

7) Let $A = \begin{bmatrix} 8 & -2 \\ 10 & -1 \end{bmatrix}$.

7a) Find the characteristic polynomial of A.

7b) What are the eigenvalues of A and their multiplicities?

7c) Find bases for each of the two eigenspaces. Indicate which basis goes with which eigenvalue.

7d) Use the answers above to diagonalize A, that is, to write $A = PDP^{-1}$ where D is a diagonal matrix.

8) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ a & b & 0 \\ b & b & 1 \end{bmatrix}$ Use simultaneuos row-reduction on A and I_3 to find A^{-1}. Under what condition(s) will A^{-1} not exist?

9) Suppose A is a 6x9 matrix with at least 3 pivot columns. What are the maximum and minimum dimensions of each of the following?

9a) Row(A)

9b) Col (A)

9c) Null (A)

9d) Rank(A)

10) Let $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 5 & 11 & 13 & 7 \\ 6 & 8 & -16 & 9 \\ 0 & 14 & 3 & 0 \end{bmatrix}$.

10a) Find $\det(A)$. Show your steps.

10b) Find $\det(A^{-1})$, or explain why it doesn't exist.

10c) Find $\det(A^2)$.