Math 205 Section B (Winter 2011)

Final Exam (60 points)

Name: Solutions.

- Check that you have 8 questions on six pages.
- Show all your work to receive full credit for a problem.

1. (7 points) Let $W = \left\{ \begin{bmatrix} p-q+r \\ r-3q \\ -2p-4q \end{bmatrix} \text{ where p, q, r are real numbers} \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^3$ by finding a spanning set for W.

$$\begin{bmatrix} p-q+r \\ r-3q \\ -2p-4q \end{bmatrix} = p\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix}.$$

So $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix} \right\}$ Since every vector in W can be written as a linear combination of these three vectors.

(b) Find two non-zero vectors in $W^{\perp}$.

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $W^{\perp}$. Then for it to be in $W^{\perp}$, it must be orthogonal to every vector in the spanning set for W found in part (a). Taking the dot product of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ with each of the vectors in the spanning set, we get the following system:

$$\begin{aligned} a - 2c &= 0 \\ a + b &= 0 \\ -a - 3b - 4c &= 0. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -3 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General soln:
$a = 2c$
$b = -2c$; c free.

Choosing c to be 1 and 2, we get two non-zero vectors in $W^{\perp}$: $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$.

2. (12 points) Let $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

(a) Find an orthogonal basis for Col A.

$\text{Col } A = \text{Span}\left\{ \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{a}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\bar{a}_2}, \underbrace{\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}}_{\bar{a}_3} \right\}$. $[\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3] \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Pivots in the first two columns of the RREF indicate that $\{\bar{a}_1, \bar{a}_2\}$ is a lin. independent set and $\bar{a}_3$ is a linear combination of $\bar{a}_1$ and $\bar{a}_2$.

Also $\bar{a}_1 \cdot \bar{a}_2 = -1 + 0 + 1 = 0$. So $\{\bar{a}_1, \bar{a}_2\}$ is an orthogonal set. Hence $\{\bar{a}_1, \bar{a}_2\}$ is an orthogonal basis for Col A.

(b) Compute the distance from $\vec{y}$ to Col A.

Distance from $\bar{y}$ to Col $A = \|\bar{y} - \hat{y}\|$, $\hat{y} = \text{proj}_{\text{Col } A} \bar{y}$.

$$\hat{y} = \left(\frac{\bar{y} \cdot a_1}{\bar{a}_1 \cdot \bar{a}_1}\right)\bar{a}_1 + \left(\frac{\bar{y} \cdot \bar{a}_2}{\bar{a}_2 \cdot \bar{a}_2}\right)\bar{a}_2 = \frac{-1}{2}\bar{a}_1 + 0 \cdot \bar{a}_2 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

$$\|\bar{y} - \hat{y}\| = \left\| \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3/2 \\ -3 \\ 3/2 \end{bmatrix} \right\|$$

$$= \sqrt{\frac{9}{4} + 9 + \frac{9}{4}}$$

$$= \frac{\sqrt{54}}{2} = \frac{3\sqrt{6}}{2}.$$

(c) Suppose $\vec{v}$ is a vector in Col A. Can the distance between $\vec{y}$ and $\vec{v}$ be less than 3? Explain.

Distance from $\bar{y}$ to Col $A = \frac{3\sqrt{6}}{2} \approx 3.67$ (as found in part (b)).

By the best approximation thm,
$\|\bar{y} - \hat{y}\| < \|\bar{y} - \bar{v}\|$ for any vector $\bar{v}$ in Col A which is not $\hat{y}$.

Thus, $3.67 < \|\bar{y} - \bar{v}\|$ if $\bar{v} \neq \hat{y}$, $\|\bar{y} - \hat{y}\| = 3.67$.
Hence, distance between $\bar{y}$ and $\bar{v}$ cannot be less than 3.

3. (6 points) A 4×4 matrix B has only two eigenvalues, 7 and 0. The dimension of the eigenspace corresponding to 7 is 2 and the dimension of the eigenspace corresponding to 0 is 1.

(a) What is rank of B? Explain.

Eigenspace corresponding to $0 = \text{Nul } B$.
So $\dim \text{Nul } B = 1$.
By the rank theorem, $4 = \dim \text{Nul } B + \text{rank } B$.
ie $4 = 1 + \text{rank } B$.
$\text{rank } B = 3$.

(b) Is $B - 7I$ invertible? Explain.

Since 7 is an eigenvalue of B,
$\det(B-7I) = 0$.
Hence $B-7I$ is not invertible.

4. (6 points) Let $\vec{v}$ be a unit vector (vector of length one) in $\mathbb{R}^8$ and let $A = \vec{v}\vec{v}^T$.

(a) Is A orthogonally diagonalizable? Explain.

$A^T = (\bar{v}\bar{v}^T)^T = (\bar{v}^T)^T \cdot \bar{v}^T = \bar{v}\bar{v}^T = A$.
So A is a symmetric matrix.
Hence it is orthogonally diagonalizable.

(b) Is $\vec{v}$ an eigenvector of A? Explain.

$A\bar{v} = (\bar{v}\bar{v}^T)\bar{v} = \bar{v}(\bar{v}^T\bar{v}) = \bar{v}(\bar{v}\cdot\bar{v})$
$= \bar{v}$ ($\bar{v}\cdot\bar{v} = 1$ since $\bar{v}$ is a unit vector.)

So $A\bar{v} = 1\cdot\bar{v}$. Hence $\bar{v}$ is an eigenvector of A.

5. (6 points) Suppose A is a 6×4 matrix such that the equation $A\vec{x} = \vec{b}$ is not consistent. The solution of the equation $A^T A\vec{x} = A^T\vec{b}$ is given below in parametric vector form.

$$\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find two least-squares solutions of $A\vec{x} = \vec{b}$.

Choosing $x_4 = 0$, $x_4 = 1$, we get the following two least-squares solutions of $A\bar{x} = \bar{b}$.

$$\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

(b) Are columns of $A^T A$ linearly independent? Explain.

Let $A^T\bar{b} = \bar{d}$. Then the eqn. $A^TA\bar{x} = \bar{d}$ is consistent and has infinitely many solns. Since the soln. set of $A^TA\bar{x} = \bar{0}$ has the same number of vectors as $A^TA\bar{x} = \bar{d}$, the eqn. $A^TA\bar{x} = \bar{0}$ has infinitely many solns. as well. So the columns of A^TA are not linearly independent.

6. (6 points) Let B be a 5×5 orthogonal matrix. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by $T(\vec{x}) = B\vec{x}$.

(a) Show that $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^5$.

$$T(\bar{x}) \cdot T(\bar{y}) = (B\bar{x}) \cdot (B\bar{y}) = (B\bar{x})^T(B\bar{y}) = (\bar{x}^TB^T)(B\bar{y})$$
$$= \bar{x}^T(B^TB)\bar{y}.$$

Since B is an orthogonal matrix, $B^{-1} = B^T$. So $B^TB = I_5$.
Hence $T(\bar{x}) \cdot T(\bar{y}) = \bar{x}^T\bar{y} = \bar{x} \cdot \bar{y}$.

(b) Is T one-to-one? Explain.

Consider the eqn. $T(\bar{x}) = \bar{0}$ i.e. $B\bar{x} = \bar{0}$.
Since B is invertible, $B \sim I_5$. So $[B\ \bar{0}] \sim [I_5\ \bar{0}]$.
Hence $B\bar{x} = \bar{0}$ has only one soln. — the trivial soln.
So T is one-to-one.

7. (8 points) Define a linear transformation $T : \mathbb{M}_{2\times 2} \to \mathbb{P}_2$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + bt + (c+d)t^2.$$

(a) Find a spanning set for the kernel (or null space) of T.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in the null space of T if

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$. i.e $a + bt + (c+d)t^2 = 0$.

So $a = 0$, $b = 0$, $c + d = 0$ ie $c = -d$.

Thus any matrix in the null space of T has the form. $\begin{bmatrix} 0 & 0 \\ -d & d \end{bmatrix} = d\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$.

So any matrix in the null space of T is a multiple of the matrix $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ and hence null space $= \text{Span}\left\{\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\right\}$.

(b) Is $\vec{p}(t) = t - 11t^2$ in the range of T? If so, find a matrix A such that $T(A) = \vec{p}(t)$. If not, explain why not.

We will try to find $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$T(A) = t - 11t^2$.

$T(A) = a + bt + (c+d)t^2$.

So $a + bt + (c+d)t^2 = t - 11t^2$.

This gives $a = 0$, $b = 1$, $c + d = -11$. Let $c = -5$, $d = -6$.

Thus, $\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$ is one matrix such that

$T\left(\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\right) = t - 11t^2$.

Hence $\vec{p}(t) = t - 11t^2$ is in the range of T.

8. (9 points) Let $\vec{p}_1(t) = 1 - 3t + 2t^2 - 4t^3$, $\vec{p}_2(t) = 2 - t + 4t^2$, $\vec{p}_3(t) = -4 - 3t^2 + 7t^3$ and $\vec{p}_4(t) = -1 - 4t + 3t^2 + 3t^3$ be polynomials in $\mathbb{P}_3$. Let $H = \text{Span}\,\{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$.

(a) Find a basis $\mathcal{B}$ for H.

Using the isomorphism from $\mathbb{P}_3 \longrightarrow \mathbb{R}^4$ given by $a+bt+ct^2+dt^3 \longmapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, we can write $\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4$ as the vectors $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 3 \\ 3 \end{bmatrix}$. $\begin{bmatrix} 1 & 2 & -4 & -1 \\ -3 & -1 & 0 & -4 \\ 2 & 4 & -3 & 3 \\ -4 & 0 & 7 & 3 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ is a lin. ind. set and $\bar{p}_4$ is a linear combination of $\bar{p}_1, \bar{p}_2, \bar{p}_3$ because of the pivots in the first three columns of the RREF.

Hence $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ is a basis for H. ie $\mathcal{B} = \{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ is a basis for H.

(b) Is $H = \mathbb{P}_3$? Explain.

As seen in part (a), $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ is a basis for H.

So $\dim H = 3$. But $\dim \mathbb{P}_3 = 4$.

Hence $H \neq \mathbb{P}_3$.

(c) Let $\vec{p}(t) = -5t - 8t^3$. Is $\vec{p}$ in H? Explain. If so, write the coordinates of $\vec{p}$ with respect to the basis $\mathcal{B}$ you found in part (a).

To decide if $\bar{p}$ is in H, we solve the eqn.

$$c_1\bar{p}_1 + c_2\bar{p}_2 + c_3\bar{p}_3 = \bar{p}.$$

We use the isomorphism described in part (a) to form the augmented matrix $\begin{bmatrix} 1 & 2 & -4 & 0 \\ -3 & -1 & 0 & -5 \\ 2 & 4 & -3 & 0 \\ -4 & 0 & 7 & -8 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So $c_1 = 2, c_2 = -1, c_3 = 0$. Since eqn. is consistent, $\bar{p}$ is in H. $[\bar{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.