Math 205 Section B (Winter 2011)

Final Exam (60 points)

Name: _____

- Check that you have 8 questions on six pages.
- Show all your work to receive full credit for a problem.

1. (7 points) Let $W = \left\{ \begin{bmatrix} p-q+r \\ r-3q \\ -2p-4q \end{bmatrix} \text{ where p, q, r are real numbers} \right\}$.

   (a) Show that W is a subspace of $\mathbb{R}^3$ by finding a spanning set for W.

   (b) Find two non-zero vectors in $W^{\perp}$.

2. (12 points) Let $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

(a) Find an orthogonal basis for Col A.

(b) Compute the distance from $\vec{y}$ to Col A.

(c) Suppose $\vec{v}$ is a vector in Col A. Can the distance between $\vec{y}$ and $\vec{v}$ be less than 3? Explain.

3. (6 points) A 4×4 matrix B has only two eigenvalues, 7 and 0. The dimension of the eigenspace corresponding to 7 is 2 and the dimension of the eigenspace corresponding to 0 is 1.

   (a) What is rank of B? Explain.

   (b) Are the columns of the matrix $B - 7I$ linearly independent? Explain.

4. (6 points) Let $\vec{v}$ be a unit vector (vector of length one) in $\mathbb{R}^8$ and let $A = \vec{v}\vec{v}^T$.

   (a) Is A orthogonally diagonalizable? Explain.

   (b) Is $\vec{v}$ an eigenvector of A? Explain.

5. (6 points) Suppose A is a 6×4 matrix such that the equation $A\vec{x} = \vec{b}$ is not consistent. The solution of the equation $A^T A\vec{x} = A^T\vec{b}$ is given below in parametric vector form.

$$\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find two least-squares solutions of $A\vec{x} = \vec{b}$.

(b) What is det $(A^T A)$? Explain.

6. (6 points) Let B be a 5×5 orthogonal matrix. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by $T(\vec{x}) = B\vec{x}$.

(a) Show that $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^5$.

(b) Is T one-to-one? Explain.

7. (8 points) Define a linear transformation $T : \mathbb{M}_{2\times 2} \to \mathbb{P}_2$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + bt + (c+d)t^2.$$

(a) Find a spanning set for the kernel (or null space) of T.

(b) Is $\vec{p}(t) = t - 11t^2$ in the range of T? If so, find a matrix A such that $T(A) = \vec{p}(t)$. If not, explain why not.

8. (9 points) Let $\vec{p}_1(t) = 1 - 3t + 2t^2 - 4t^3$, $\vec{p}_2(t) = 2 - t + 4t^2$, $\vec{p}_3(t) = -4 - 3t^2 + 7t^3$ and $\vec{p}_4(t) = -1 - 4t + 3t^2 + 3t^3$ be polynomials in $\mathbb{P}_3$. Let $H = \text{Span}\,\{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$.

   (a) Find a basis $\mathcal{B}$ for H.

   (b) Is $H = \mathbb{P}_3$? Explain.

   (c) Let $\vec{p}(t) = -5t - 8t^3$. Is $\vec{p}$ in H? Explain. If so, write the coordinates of $\vec{p}$ with respect to the basis $\mathcal{B}$ you found in part (a).