Math 205A Final Exam (65 points)

Name: ______________________________

- Check that you have 7 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (7 points) Let A be a 4×3 matrix, B be a 3×4 matrix and C be a 4×4 matrix, with $\det(AB) = 2$ and $\det C = -10$.

   (a) Find $\det ABC$. Is ABC an invertible matrix? Explain.

   (b) Find $\det B^T A^T$. Explain.

2. (12 points) Let $\vec{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

(a) Is $\{\vec{u}_1, \vec{u}_2\}$ an orthogonal basis for W? Explain.

(b) Compute the distance from $\vec{y}$ to W.

Problem 2 continued from previous page. In parts (c) and (d) below, W and $\vec{y}$ are the same as on the previous page.

(c) Suppose $\vec{v}$ is a vector in $\mathbb{R}^3$ such that the distance between $\vec{y}$ and $\vec{v}$ is 1. Can $\vec{v}$ be in W? Explain.

(d) In your computation in part (b), did you find a vector in $W^{\perp}$? If so, what is that vector?

3. (6 points) $H = \{\text{all polynomials } \vec{\mathrm{p}}(\mathrm{t}) \text{ in } \mathbb{P}_2 \text{ such that } \vec{\mathrm{p}}(0) = 0\}$. Is H a subspace of $\mathbb{P}_2$? Explain.

4. (12 points) Define a linear transformation $T : \mathbb{M}_{2\times 2} \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d \\ b+c \end{bmatrix}.$$

(a) Find a set of matrices that spans the kernel (or null space) of T.

(b) The kernel of T is a subspace of $\mathbb{M}_{2\times 2}$. Use the spanning set you found in part (a) to find a basis for the kernel of T.

(c) Is T one-to-one? Explain.

Problem 4 continued from previous page. In part (d) below, the linear transformation T is the same as on the previous page.

(d) Is $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$ in the range of T? If so, find a matrix A such that $T(A) = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$. If not, explain why not.

5. (6 points) Let $\vec{p}_1(t) = 2 - t$ and $\vec{p}_2(t) = 7t$ be polynomials in $\mathbb{P}_1$.

(a) Verify that $\mathcal{B} = \{\vec{p}_1, \vec{p}_2\}$ is a basis for $\mathbb{P}_1$ by showing that the set satisfies the two conditions in the definition of a basis.

(b) Find the polynomial $\vec{q}$ such that $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

6. (12 points) A 3×3 matrix A has only two eigenvalues, -1 and 5. A basis for the eigenspace corresponding to -1 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and a basis for the eigenspace corresponding to 5 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Let $\vec{x} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$

(a) Find $A\vec{x}$.

(b) What is dimension of Nul $A + I$? What is rank of $A + I$? Explain.

Problem 6 continued from previous page. In parts (c) and (d) below, the matrix A is the same as on the previous page.

(c) Are the columns of the matrix $A - 5I$ linearly independent? Explain.

(d) Is A diagonalizable? Explain.

7. (10 points) The length of a spring changes when we apply a force to it. Hooke's law tells us that the force (f) and the length (l) are related by the equation $l = a + bf$ where a and b are constants that depend on the spring. You would like to find these constants for a particular spring. To this end you collect the following experimental data by suspending a weight (which gives the force f in ounces) from the spring and then measuring the length l (in inches) of the spring.

f	2	4	6	8
l	8.2	11.6	14.3	17.5

Find a and b so that the Hooke's law equation is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine a and b.)