1. (10 points) Let \( A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \). Use this matrix to answer the following questions:

(a) Find a basis for \( \text{Col} \ A \). What is \( \dim \text{Col} \ A \)?

\[
A \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Pivots in first two columns.
Basis for \( \text{Col} \ A = \left\{ \begin{bmatrix} 1/3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 5/3 \\ 1 \end{bmatrix} \right\} \quad \dim \text{Col} \ A = 2.

(b) Write one of the columns of \( A \) as a linear combination of the other two columns. (You can use your answer in part (a) to decide which column to write as a linear combination of the other two.)

\[
A \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(Here, think about \( A \) as the augmented matrix for the system \( B \vec{x} = \vec{b} \), where \( B = \begin{bmatrix} 1/3 & -2/3 \\ -2/3 & 5/3 \\ 1/3 & 1 \end{bmatrix} \) and \( \vec{b} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \)).

This gives \( \vec{c}_3 = 3 \vec{c}_1 + \vec{c}_2 \).

(c) Is the equation \( A\vec{x} = \vec{b} \) consistent for every choice of \( \vec{b} \)? Explain.

Since the reduced echelon form of \( A \) does not have a pivot in each row, the equation \( A\vec{x} = \vec{b} \) is not consistent for every choice of \( \vec{b} \).
2. (10 points) Let \( \bar{p}_1(t) = 1 - t^2, \bar{p}_2(t) = t - t^2, \bar{p}_3(t) = 2 - 2t + t^2 \).

(a) Use coordinate vectors to show that \( B = \{ \bar{p}_1, \bar{p}_2, \bar{p}_3 \} \) is a basis for \( \mathbb{P}_2 \). Find the polynomial \( q \) in \( \mathbb{P}_2 \), given that \( [q]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \).

\[
\overline{p}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overline{p}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \overline{p}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \text{(w.r.t the standard basis \{1, t, t^2\} for \( \mathbb{P}_2 \))}
\]
\[
\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{So } B \text{ is a basis for } \mathbb{P}_2.
\]
\[
q = \overline{p}_1 - 2\overline{p}_2 + \overline{p}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 + 2 \end{bmatrix}
\]
So \( \overline{q}(t) = -3 - 4t + 2t^2 \).

(b) Define a linear transformation \( T : \mathbb{P}_2 \to \mathbb{R}^2 \) by \( T(a_0 + a_1t + a_2t^2) = \begin{bmatrix} a_0 \\ a_1 + a_2 \end{bmatrix} \). Find \( T(\bar{p}_1), T(\bar{p}_2), \) and \( T(\bar{p}_3) \).

\[
T(\bar{p}_1) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix},
\]
\[
T(\bar{p}_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
\[
T(\bar{p}_3) = \begin{bmatrix} 2 \\ -2 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\]

(c) For the linear transformation \( T \) defined in part (b), the kernel of \( T \) is a subspace of \( \mathbb{P}_2 \) of dimension one. Find a basis for the kernel of \( T \). (Your answer to part (b) can help you to find a basis.) Is \( T \) one-to-one? Explain.

We know \( T(\bar{p}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(from (b))} \).

So \( \bar{p}_2 \) is in kernel of \( T \).

Since \( \ker T \) is of dim one, basis for \( \ker T = \{ \bar{p}_2 \} \).

The equation \( T(\bar{x}) = \overline{0} \) has a non-trivial solution because \( \bar{p}_2 \) is one solution of this equation.

So \( T \) is not one-to-one.
3. (10 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and dimension of the subspace.

(a) All polynomials in \( \mathbb{P}_1 \) such that \( \bar{p}(1) = 1 \).

Let \( \bar{0} \) be the zero polynomial.

Then \( \bar{0}(1) = 0 \).

So \( \bar{0} \) is not in the set.

So the set is not a subspace.

(b) \( W = \{ \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} : a, b \text{ in } \mathbb{R} \} \).

\[
\begin{bmatrix}
   a+b & 0 \\
   0 & a-b
\end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

So \( W = \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \).

If we consider these two matrices as vectors in \( \mathbb{R}^4 \), then they form a linearly independent set (as there is a pivot in each column).

So basis for \( W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \).

\( \dim W = 2 \).
4. (12 points) Let \( W \) be the subspace spanned by the two vectors \( \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \).

(a) Is \( \{\vec{u}_1, \vec{u}_2\} \) an orthogonal basis for \( W \)? Explain.

\[
\vec{u}_1 \cdot \vec{u}_2 = 6 - 3 - 3 = 0 \quad \text{so} \quad \{\vec{u}_1, \vec{u}_2\} \text{ is an orthogonal set.}
\]

An orthogonal set is linearly independent. Also, the set \( \{\vec{u}_1, \vec{u}_2\} \) spans \( W \).

So \( \{\vec{u}_1, \vec{u}_2\} \) is a basis for \( W \). Thus it is an orthogonal basis.

(b) Let \( \vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \). Find a vector in \( W \) that is closest to \( \vec{y} \) and then find the distance between \( \vec{y} \) and \( W \).

Vector in \( W \) that is closest to \( \vec{y} \) is given by

\[
\frac{\vec{y}}{\vec{y}} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{0}{1} \vec{u}_1 + \frac{6 + 3}{36 + 9 + 9} \vec{u}_2 = \frac{1}{6} \vec{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}
\]

Distance between \( \vec{y} \) and \( W \) = distance between \( \vec{y} \) and \( \frac{\vec{y}}{\vec{y}} \)

\[
= \left\| \vec{y} - \frac{\vec{y}}{\vec{y}} \right\| = \left\| \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \right\| = \sqrt{0 + \frac{1}{4} + \frac{1}{4}} = \frac{1}{2}
\]

(c) Find a vector in \( W^\perp \).

Vector in \( W^\perp = \vec{y} - \frac{\vec{y}}{\vec{y}} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \)

(d) Find a basis and dimension of \( W^\perp \). (Hint: Think geometrically and use your answer to part (c).)

\( \{\vec{u}_1, \vec{u}_2\} \) is a basis for \( W \) so \( \dim W = 2 \).

\( W \) is a subspace of dimension 2 in \( \mathbb{R}^3 \). So it is a plane in \( \mathbb{R}^3 \). \( W^\perp \) is then a line perpendicular to this plane in \( \mathbb{R}^3 \). So \( \dim W^\perp = 1 \).

Basis = \( \{ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \} \) (Any vector in \( W^\perp \) will)
5. (10 points) A quadratic form on \( \mathbb{R}^3 \) is given by the formula \( Q(\vec{x}) = 7x_1^2 - 8x_1x_2 + 8x_1x_3 + 5x_2^2 + 9x_3^2 \).

(a) Find the symmetric matrix \( A \) of the quadratic form.

\[
A = \begin{bmatrix}
7 & -4 & 4 \\
-4 & 5 & 0 \\
4 & 0 & 9
\end{bmatrix}
\]

(b) Find a matrix \( P \) such that the change of variable \( \vec{x} = P\vec{y} \) transforms the quadratic form into one with no cross-product term. (You may use the fact that the eigenvalues of \( A \) are 13, 7, and 1.

For \( \lambda = 13 \): \( A - 13I = \begin{bmatrix}
-6 & 4 & 4 \\
4 & -8 & 0 \\
4 & 0 & -4
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 0
\end{bmatrix} \) Basis for eigenspace: \{ \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} \} or \{ \begin{bmatrix} \frac{1}{2} \\ -1/2 \\ 0 \end{bmatrix} \}.

For \( \lambda = 7 \): \( A - 7I = \begin{bmatrix}
0 & -4 & 4 \\
-4 & -2 & 0 \\
4 & 0 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 1/2 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \) Basis = \{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}.

For \( \lambda = 1 \): \( A - I = \begin{bmatrix}
6 & -4 & 4 \\
-4 & 4 & 0 \\
4 & 0 & 8
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \) Basis = \{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \}.

\[
\begin{bmatrix}
2 & -1 & -2 \\
-1 & 2 & -2 \\
2 & 2 & 1
\end{bmatrix}
\]
Length of each column = \( \sqrt{4+4+1} = 3 \) So \( P = \begin{bmatrix}
2/3 & -1/3 & -2/3 \\
-1/3 & 2/3 & -2/3 \\
2/3 & 2/3 & 1/3
\end{bmatrix}
\]

(c) Write the new quadratic form with no cross-product term. Is \( Q \) positive definite, negative definite or indefinite? Explain.

Use eigenvalues of \( A \) to write the transformed quadratic form: 13\( x_1^2 \) + 7\( x_2^2 \) + \( x_3^2 \).

Since all the eigenvalues of \( A \) are positive, \( Q \) is positive definite.
6. (6 points) For a diagonalizable matrix $A$, show that $\det A$ is the product of the eigenvalues of $A$. (Hint: Since $A$ is diagonalizable, you can write $A$ as the product of certain matrices. Then take the determinant of both sides of the equation.)

$$A = PDP^{-1} \quad \Rightarrow \quad \det A = \det (PDP^{-1}) = \det P \det D \det P^{-1}$$

$$\quad = \det D \det P \det P^{-1}$$

$$\quad = \det D \det (PP^{-1})$$

$$\quad = \det D \det I$$

$$\quad = \det D \cdot (\text{since } \det I = 1).$$

So $\det A = \det D$. $D$ is a diagonal matrix, so $\det D$ is the product of the diagonal entries of $D$ and these entries are the eigenvalues of $A$.

7. (6 points) Let $S = \{\vec{u}_1, \vec{u}_2\}$ be an orthonormal set in $\mathbb{R}^2$ and let $A = [\vec{u}_1 \, \vec{u}_2]$ be the matrix whose columns are the vectors in $S$.

(a) Show that the set $S$ is linearly independent using the definition of linear independence. (Hint: Show that the equation $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ has only the trivial solution. For example, to show $c_1 = 0$, do the dot product of each side of the equation with $\vec{u}_1$.)

$$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$$

$$\vec{u}_1 \cdot (c_1\vec{u}_1 + c_2\vec{u}_2) = \vec{u}_1 \cdot \vec{0} = 0$$

$$c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\vec{u}_1 \cdot \vec{u}_2) = 0$$

$$c_1(\vec{u}_1 \cdot \vec{u}_1) = 0 \quad (\text{since } \vec{u}_1 \cdot \vec{u}_1 = 0 \text{ as } \{\vec{u}_1, \vec{u}_2\} \text{ is an orthogonal set})$$

$$c_1 = 0 \quad (\text{since } \vec{u}_1 \cdot \vec{u}_1 = 1 \text{ as } \vec{u}_1 \text{ is of unit length})$$

(b) Is $A$ invertible? Explain.

Columns of $A$ are linearly independent, as shown in part (a). So $A$ has a pivot in each column. Since $A$ is a $2 \times 2$ matrix, we then have a pivot in each row. So $A \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $A$ is invertible.
8. (7 points) Suppose $A$ is a $4 \times 6$ matrix.

(a) Find $k$ such that $\text{Col} \ A$ is a subspace of $\mathbb{R}^k$.

$$k = 4.$$

(b) Is it possible that $\dim \text{Nul} \ A = 1$? Explain. (Hint: use the rank theorem and your answer in part (a).)

By the rank theorem,

$$6 = \text{rank} \ A + \dim \text{Nul} \ A$$

Since $\text{Col} \ A$ is a subspace of $\mathbb{R}^4$, rank $A$ is at most 4.

so $\text{Nul} \ A$ has dimension at least 2.

so $\dim \text{Nul} \ A \neq 1$.

9. (14 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

(a) If 4 is an eigenvalue of a matrix $A$, then what is $\det (A - 4I)$?

$$\det (A - 4I) = 0.$$

(b) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is an onto linear transformation. What is the dimension of range of $T$?

$$\dim \text{range} \ T = 3.$$

(c) For an orthogonal matrix $U$, what is the inverse of $U$?

$$U^T$$

(d) Suppose $B$ is a $5 \times 5$ matrix with $\det B = 10$. What is $\det 2B$?

$$\det 2B = 2^5 \det B = 2^5 \cdot 10 = 320.$$
(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 - 5x_2, 2x_3)$. What is the standard matrix of $T$?

$$
\begin{bmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{bmatrix}
= \begin{bmatrix}
1 & -5 & 0 \\
0 & 0 & 2
\end{bmatrix}.
$$

(f) What is the length of the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$?

$$
\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}.
$$

(g) Let $A = PDP^{-1}$ where $P = \begin{bmatrix} 0 & -30 & 39 & 11 \\ -3 & -7 & 5 & -3 \\ 3 & 3 & 0 & 4 \\ 2 & 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Write the eigenvalues of $A$ and a basis for the eigenspace of each eigenvalue.

**Eigenvalues of $A$:** 2, 4, 5

**Basis for eigenspace:**

$$
\lambda = 2 : \begin{bmatrix} 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \lambda = 4 : \begin{bmatrix} -30 \\ -7 \\ 3 \\ 0 \end{bmatrix}, \quad \lambda = 5 : \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.
$$

(h) What is the characteristic polynomial of the matrix $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$?

$$
\det \begin{bmatrix} 3 - \lambda & -1 \\ 1 & 5 - \lambda \end{bmatrix} = (3 - \lambda)(5 - \lambda) + 1
= \lambda^2 - 8\lambda + 16.
$$