Math 205 Section B
Final Exam (85 points)

Name: ____

- Check that you have 9 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (10 points) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$. Use this matrix to answer the following questions:

   (a) Find a basis for Col A. What is dim Col A?

   (b) Write one of the columns of A as a linear combination of the other two columns. (You can use your answer in part (a) to decide which column to write as a linear combination of the other two.)

   (c) Is the equation $A\vec{x} = \vec{b}$ consistent for every choice of $\vec{b}$? Explain.

2. (10 points) Let $\vec{p_1}(t) = 1 - t^2$, $\vec{p_2}(t) = t - t^2$, $\vec{p_3}(t) = 2 - 2t + t^2$.

(a) Use coordinate vectors to show that $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for $\mathbb{P}_2$. Find the polynomial $\vec{q}$ in $\mathbb{P}_2$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(b) Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_1 + a_2 \end{bmatrix}$. Find $T(\vec{p_1})$, $T(\vec{p_2})$, and $T(\vec{p_3})$.

(c) For the linear transformation T defined in part (b), the kernel of T is a subspace of $\mathbb{P}_2$ of dimension one. Find a basis for the kernel of T. (Your answer to part (b) can help you to find a basis.) Is T one-to-one? Explain.

3. (10 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and dimension of the subspace.

   (a) All polynomials in $\mathbb{P}_1$ such that $\vec{p}(1) = 1$.

   (b) $W = \left\{ \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$.

4. (12 points) Let W be the subspace spanned by the two vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$.

(a) Is $\{\vec{u}_1, \vec{u}_2\}$ an orthogonal basis for W? Explain.

(b) Let $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find a vector in W that is closest to $\vec{y}$ and then find the distance between $\vec{y}$ and W.

(c) Find a vector in $W^{\perp}$.

(d) Find a basis and dimension of $W^{\perp}$. (Hint: Think geometrically and use your answer to part (c).)

5. (10 points) A quadratic form on $\mathbb{R}^3$ is given by the formula $Q(\vec{x}) = 7x_1^2 - 8x_1x_2 + 8x_1x_3 + 5x_2^2 + 9x_3^2$.

(a) Find the symmetric matrix A of the quadratic form.

(b) Find a matrix P such that the change of variable $\vec{x} = P\vec{y}$ transforms the quadratic form into one with no cross-product term. (You may use the fact that the eigenvalues of A are 13, 7, and 1.

(c) Write the new quadratic form with no cross-product term. Is Q positive definite, negative definite or indefinite? Explain.

6. (6 points) For a diagonalizable matrix A, show that $\det A$ is the product of the eigenvalues of A. (Hint: Since A is diagonalizable, you can write A as the product of certain matrices. Then take the determinant of both sides of the equation.)

7. (6 points) Let $S = \{\vec{u}_1, \vec{u}_2\}$ be an orthonormal set in $\mathbb{R}^2$ and let $A = [\vec{u}_1 \;\; \vec{u}_2]$ be the matrix whose columns are the vectors in S.

   (a) Show that the set S is linearly independent using the definition of linear independence. (Hint: Show that the equation $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$ has only the trivial solution. For example, to show $c_1 = 0$, do the dot product of each side of the equation with $\vec{u}_1$.)

   (b) Is A invertible? Explain.

8. (7 points) Suppose A is a 4×6 matrix.

   (a) Find k such that Col A is a subspace of $\mathbb{R}^k$.

   (b) Is it possible that dim Nul $A = 1$? Explain. (Hint: use the rank theorem and your answer in part (a).)

9. (14 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

   (a) If 4 is an eigenvalue of a matrix A, then what is det $(A - 4I)$?

   (b) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is an onto linear transformation. What is the dimension of range of T?

   (c) For an orthogonal matrix U, what is the inverse of U?

   (d) Suppose B is a 5×5 matrix with det $B = 10$. What is det $2B$?

(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 - 5x_2, 2x_3)$. What is the standard matrix of T?

(f) What is the length of the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$?

(g) Let $A = PDP^{-1}$ where $P = \begin{bmatrix} 0 & -30 & 39 & 11 \\ -3 & -7 & 5 & -3 \\ 3 & 3 & 0 & 4 \\ 2 & 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Write the eigenvalues of A and a basis for the eigenspace of each eigenvalue.

(h) What is the characteristic polynomial of the matrix $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$?