Final Exam, Math 205A (Linear Algebra)

This take-home exam is due by 5 PM on **Friday, April 12**. You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the appropriate place on the other side of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Matrix multiplications and reduced row echelon forms may be done on MATLAB or a calculator, but please show all other work.

1. (15 points) Find the complete solution of the system $\begin{pmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ -3 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ -3 \\ -7 \end{pmatrix}$.

2. (27 points) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 8 & -2 \\ 1 & -2 & 7 \end{pmatrix}$.

(a) Find the LU factorization of A.

(b) Find the determinant of L and the determinant of U.

(c) Use your answer to (a) to solve $A\vec{x} = \begin{pmatrix} 7 \\ -14 \\ 47 \end{pmatrix}$.

(d) Use your answer to (b) to find the determinant of A.

(e) Find L^{-1} and U^{-1}.

(f) Use your answer to (e) to find A^{-1}.

3. (18 points) Find a basis for each of the four subspaces associated with the matrix

$$B = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 2 \\ 1 & 3 & 2 & 7 \end{pmatrix}$$

What is the factored form of B that displays these bases?

4. (10 points) Your column space basis in problem 3 should consist of 2 vectors in $\mathbb{R}^3$. What is their cross product? Is there anything interesting about the answer? Explain.

5. (18 points) Explain how you can tell that

$$P = \frac{1}{6} \begin{pmatrix} 4 & 0 & -2 & 0 & -2 \\ 0 & 3 & 0 & -3 & 0 \\ -2 & 0 & 4 & 0 & -2 \\ 0 & -3 & 0 & 3 & 0 \\ -2 & 0 & -2 & 0 & 4 \end{pmatrix}$$

is a projection matrix. Find a basis for the subspace S of $\mathbb{R}^5$ that P projects onto, and a basis for $S^{\perp}$.

6. (20 points) Let $C = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$.

(i) Find the eigenvalues λ_1 and λ_2 of C, and the corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

(ii) Let $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and let E be the matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$. Check that $CE = E\Lambda$.

(iii) Find E^{-1}.

(iv) Use Λ, E and E^{-1} to obtain a formula for C^k, for a generic power k. For which powers is the formula valid?

7. (22 points) Suppose S is an s-dimensional subspace of $\mathbb{R}^n$, T is a t-dimensional subspace of $\mathbb{R}^n$, and every vector in S is perpendicular to every vector in T. Let P_s be the matrix that projects vectors in $\mathbb{R}^n$ onto S, and let P_t be the matrix that projects vectors in $\mathbb{R}^n$ onto T. Finally, let U be the set of all vectors in $\mathbb{R}^n$ which are linear combinations of the vectors in S and the vectors in T.

(i) What is the dimension of U? Explain.

(ii) Explain why $P_s + P_t$ is symmetric.

(iii) What is $P_s P_t$? (Hint: what does P_s do to a vector in T?) What is $P_t P_s$?

(iv) Simplify $(P_s + P_t)^2$ as much as you can. What do you conclude?

(v) What is the trace of $P_s + P_t$? How do you know?

(vi) Explain why the column space of $P_s + P_t$ must be contained in U.

(vii) Explain why all this implies that $P_s + P_t$ is the projection matrix onto U.

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________