NAME: KEY

Math 106B - Final Exam, April 11, 2007

INSTRUCTIONS: Show all of your work and justify your solutions. Circle your final answers. Cross out any unnecessary work. The last page of this exam contains formulæ which may or may not be helpful in various problems.

1. (7 points) Consider the integral $I = \int_0^3 e^{-x^2}\,dx$. Using a right-endpoint sum to approximate I, how many intervals are necessary to guarantee accuracy to within ± 0.005?

For R_n, we have $|I - R_n| \leq \frac{K_1 (b-a)^2}{2n}$. So we need K_1.

$|f'(x)| \leq K_1$ for all x in $[0,3]$.

$f(x) = e^{-x^2}$, so $f'(x) = -2x e^{-x^2}$. Looking at a graph of

$y = |-2xe^{-x^2}|$, we see the max value is approximately $0.858\ldots$

for x-values in $[0,3]$. So we can let $K_1 = 0.86$, meaning $|f'(x)| \leq K_1$.

Then $|I - R_n| \leq \frac{K_1 (b-a)^2}{2n} = \frac{0.86(3-0)^2}{2n} = \frac{3.87}{n}$.

We want this less than 0.005. So $\frac{3.87}{n} < 0.005$

So $774 < n$.

Hence, we want at least $\boxed{775 \text{ intervals}}$ to guarantee accuracy within ± 0.005 of I.

2. (8 points) Solve the initial value problem $y' = \frac{1+y^2}{x}$, $y(e) = 0$.

$$\frac{dy}{dx} = \frac{1+y^2}{x} \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{x} \Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{x}.$$

So $\arctan y = \ln x + C$. Using the fact that $y(e) = 0$, we get

$\arctan 0 = \ln e + C$

$0 = 1 + C$

So $C = -1$.

$\arctan y = \ln x + -1$

$\boxed{\text{So } y = \tan((\ln x) - 1).}$

Alternatively, if $\arctan y = \ln x + c$, then $y = \tan((\ln x) + c)$, and since $y(e) = 0$,

$0 = \tan((\ln e) + c)$

$0 = \tan(-1 + C)$

So $C = 1$ or $1 + \pi$ or $1 + 2\pi$ or ... would work.

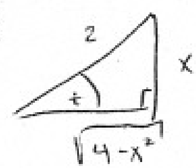
$\rightarrow \boxed{y = \tan((\ln x) - 1)}.$

3. (10 points) Evaluate $\int \frac{dx}{(4-x^2)^{3/2}}$. (Hint: Use a trig substitution.)

Note: $(4-x^2)^{3/2} = (\sqrt{4-x^2})^3$.

$a^2 - x^2$ form with $a = 2$. So let $x = 2\sin t$

$dx = 2\cos t\,dt$. $\rightarrow \frac{x}{2} = \sin t$



$$\int \frac{dx}{(4-x^2)^{3/2}} = \int \frac{2\cos t\,dt}{(4-4\sin^2 t)^{3/2}}$$

$$= 2\int \frac{\cos t\,dt}{(\sqrt{4\cos^2 t})^3}$$

$$= 2\int \frac{\cos t\,dt}{(2\cos t)^3}$$

$$= 2\int \frac{\cos t\,dt}{8\cos^3 t}$$

$$= \frac{1}{4}\int \frac{1}{\cos^2 t}\,dt$$

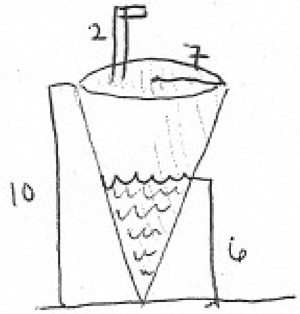
$$= \frac{1}{4}\int \sec^2 t\,dt$$

$$= \frac{1}{4}\tan t + C. \quad \left(\frac{d}{dt}\tan t = \sec^2 t\right)$$

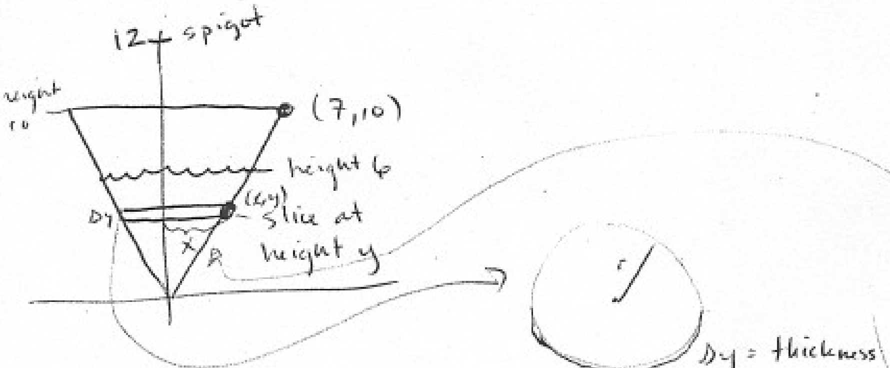
$$\boxed{= \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C}$$

↑ using triangle.

4. (10 points) A conical tank (think ice cream cone with its point at the bottom) has height 10 feet. At the top of the conical tank, the radius is 7 feet. The tank contains water that is 6 feet deep at its deepest point. Set up an integral to calculate the amount of work to pump the water out of the tank to a spigot that is 2 feet above the top of the tank. (Water weighs 62.4 pounds per cubic foot.)



Cross section:



Radius of slice at level y is x, the dist from y-axis to the pt on the graph of the line.

$V_{slice} = \pi x^2 \Delta y$. Need x^2 in terms of y. Pt (x,y) is on a line - get eq'n of line.

Slope $= \frac{\Delta y}{\Delta x} = \frac{10-0}{7-0} = \frac{10}{7}$. y-int $= 0$.

$y = \frac{10}{7}x + 0$

So $x = \frac{7}{10}y$.

$V_{slice} = \pi\left(\frac{7}{10}y\right)^2 \Delta y$. (ft^3).

$F_{slice} = 62.4\pi\left(\frac{7}{10}y\right)^2 \Delta y$ $\left(ft^3 \cdot \frac{lb}{ft^3} = lb\right)$

Move slice from level y to level 12. Dist $= 12-y$.

Work for slice $= F \cdot d = 62.4\pi\left(\frac{7}{10}y\right)^2 \Delta y\,(12-y)$.

$$\boxed{\text{Total work} = \int_{y=0}^{y=6} 62.4\pi\left(\frac{7}{10}y\right)^2(12-y)\,dy.}$$

5. (12 points) Let $f(x) = \sqrt{x}$.

(a) Find $P_2(x)$, the degree-2 Taylor polynomial of $f(x)$ centered at $x = 1$.

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$ $\quad f''(1) = -\frac{1}{4}$

$$P_2(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2$$

$$P_2(x) = 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2}(x-1)^2$$

so $\boxed{P_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.}$

(b) Use $P_2(x)$ to approximate $\sqrt{2}$. (Write your answer as a fraction instead of a decimal approximation.)

$P_2(x) \approx \sqrt{x}$, so $P_2(2) \approx \sqrt{2}$.

$$P_2(2) = 1 + \frac{1}{2}(2-1) - \frac{1}{8}(2-1)^2 = 1 + \frac{1}{2} - \frac{1}{8} = \boxed{\frac{11}{8}}$$

(c) According to Taylor's Theorem, what is the maximum approximation error committed by $P_2(x)$ on the interval $[1, 2]$?

$|f'''(x)| \le K_3$ for all x in $[1,2]$.

$f'''(x) = \frac{3}{8} x^{-5/2} = \frac{3}{8x^{5/2}}$. This function is decreasing and positive, so in absolute value, it is largest when x is smallest in $[1,2]$, hence at $x = 1$.

So $|f'''(x)| \le \frac{3}{8}$. Let $K_3 = \frac{3}{8}$.

Then $|\sqrt{x} - P_2(x)| \le \frac{K_3}{3!}|x-1|^3 = \frac{3/8}{6}|x-1|^3 = \frac{1}{16}|x-1|^3$. For x-values in $[1,2]$, $|x-1| \le 1$, so $|x-1|^3 \le 1$,

so $\frac{1}{16}|x-1|^3 \le \frac{1}{16}$, so

$\boxed{|\sqrt{x} - P_2(x)| \le \frac{1}{16}}$ = max possible error.

(d) Are your answers to (b) and (c) consistent? Explain.

$\sqrt{2} \approx 1.414$. $P_2(2) = \frac{11}{8} = 1.375$.

$\sqrt{2} - P_3(2) \approx 0.039\ldots$,

which is less than $\frac{1}{16} = 0.0625$.

So Taylor's theorem is consistent with our result.

6. (8 points) Determine if the following series converges absolutely, converges conditionally, or diverges:

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}.$$

A.S.T.

$c_k = \frac{1}{k\ln k}$ ① As k increases, k inc and lnk inc,

So klnk inc, so $\frac{1}{k\ln k}$ dec.

So $c_2 > c_3 > c_4 > \dots > 0$. ✓

② $\lim_{k\to\infty} c_k = \lim_{k\to\infty} \frac{1}{k\ln k} = 0$. ✓

So, by Alt. Series Test, this series converges.

Abs conv? $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$.

Int test. $f(x) = \frac{1}{x\ln x}$. Pos, dec. ✓

$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}\,dx$. Let $u = \ln x$, $du = \frac{1}{x}dx$

$= \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{u}\,du = \lim_{t\to\infty} \ln u \Big]_{x=2}^{x=t}$

$= \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2) = \infty$.

Diverges, so series diverges,

So original series is conditionally convergent.

7. (12 points) Determine if the following series converge or diverge.

(a) $\sum_{k=0}^{\infty} \frac{k^2+4}{100k^2+3k+5}$.

$a_k = \frac{k^2+4}{100k^2+3k+4} = \frac{1+\frac{4}{k^2}}{100+\frac{3}{k}+\frac{4}{k^2}}$. $\lim_{k\to\infty} a_k = \frac{1+0}{100+0+0} = \frac{1}{100}$.

Terms don't go to 0, so, by "nth term test," the series diverges.

(b) $\sum_{k=0}^{\infty} \frac{5^k k!}{(2k)!}$. Ratio test:

$a_k = \frac{5^k k!}{(2k)!}$. $a_{k+1} = \frac{5^{k+1}(k+1)!}{(2(k+1))!} = \frac{5^{k+1}(k+1)!}{(2k+2)!}$

$\frac{a_{k+1}}{a_k} = \frac{5^{k+1}(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{5^k \cdot k!} = \frac{5(k+1)}{(2k+2)(2k+1)} = \frac{5(k+1)}{2(k+1)(2k+1)} = \frac{5}{2(2k+1)}$

So $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{5}{2(2k+1)}\right| = 0$. $0<1$, so the series converges.

8. (7 points) Find the first four non-zero terms of the power series centered at $x = 0$ for the function

$$f(x) = \frac{e^{2x}}{1-x}.$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so $e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$

$= 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots$

$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

So $\frac{e^{2x}}{1-x} = e^{2x}\left(\frac{1}{1-x}\right) = \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$

$= 1 + x(1\cdot 1 + 2\cdot 1) + x^2(1\cdot 1 + 2\cdot 1 + 2\cdot 1) + x^3(1\cdot 1 + 2\cdot 1 + 2\cdot 1 + \frac{4}{3}\cdot 1) + \cdots$

$\boxed{= 1 + 3x + 5x^2 + \frac{19}{3}x^3 + \cdots}$

9. (11 points) Find the interval of convergence for the power series

$$S(x) = \sum_{k=1}^{\infty} \frac{(x-3)^k}{k 2^k}.$$

Ratio test: $a_{k+1} = \frac{(x-3)^{k+1}}{(k+1)2^{k+1}}$. $a_k = \frac{(x-3)^k}{k 2^k}$.

$\frac{a_{k+1}}{a_k} = \frac{(x-3)^{k+1}}{(k+1)2^{k+1}} \cdot \frac{k 2^k}{(x-3)^k} = \frac{(x-3)\cdot k}{(k+1)\cdot 2}$

So $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$

$= \lim_{k\to\infty}\left|\frac{(x-3)\cdot k}{2(k+1)}\right|$

$= \frac{|x-3|}{2} \cdot \lim_{k\to\infty}\left|\frac{k}{k+1}\right| \cdot \frac{1/k}{1/k}$

$= \frac{|x-3|}{2} \lim_{k\to\infty}\left|\frac{1}{1+1/k}\right|$

$= \frac{|x-3|}{2} \cdot \left|\frac{1}{1+0}\right|$

$= \frac{|x-3|}{2}$.

Want $\frac{|x-3|}{2} < 1$, so $\boxed{|x-3| < 2}$

Interval (so far) is $(1,5)$. Check endpts.

$x=1$: $\sum_{k=1}^{\infty} \frac{(1-3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ = negative of alternating harmonic series, which converges.

$x=5$: $\sum_{k=1}^{\infty} \frac{(5-3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{2^k}{k2^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ = harmonic, which diverges.

$\rightarrow \boxed{\text{Interval} = [1,5)}$

10. (15 points)

(a) Evaluate $\lim_{x\to 0} \dfrac{x - \frac{x^3}{3} - \arctan x}{x^5}$.

$$= \lim_{x\to 0} \frac{(\cancel{x} - \cancel{\frac{x^3}{3}}) - (\cancel{x} - \cancel{\frac{x^3}{3}} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots)}{x^5}$$

$$= \lim_{x\to 0} \frac{-\frac{x^5}{5} + \frac{x^7}{7} - \cdots}{x^5} = \lim_{x\to 0} \frac{-\frac{1}{5} + \frac{x^2}{7} - \frac{x^4}{9} + \cdots}{1}$$

$$= -\frac{1}{5} + 0 - 0 + \cdots$$

$$\boxed{= -1/5}$$

(b) Suppose $g(x) = \arctan(x)$.

i. Evaluate $g^{(99)}(0)$.

$g(x) = \arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, 99th deriv only depends on x^{99} term.

(or, $g^{(99)}(0) = 99! \cdot a_{99}$.)

$x^{2k+1} = x^{99}$? $\to k = 49$. x^{99} term is

$\frac{(-1)^{49} x^{99}}{99}$, so 99th deriv is $\frac{(-1)^{49} \cdot 99!}{99}$

$\boxed{= (-1)\, 98! = g^{(99)}(0).}$

ii. Evaluate $g^{(100)}(0)$.

Look at x^{100} term.

$x^{100} = x^{2k+1} \to k = 49.5$. No such term in our sum, so we have $0\, x^{100}$. The 100th derivative of this is 0.

$\to \boxed{g^{(100)}(0) = 0.}$

1	2	3	4	5	6	7	8	9	10	TOTAL

Potentially useful formulæ

- $|I - L_n| \leq \dfrac{K_1(b-a)^2}{2n}$.
- $|I - R_n| \leq \dfrac{K_1(b-a)^2}{2n}$.
- $|I - T_n| \leq \dfrac{K_2(b-a)^3}{12n^2}$.
- $|I - M_n| \leq \dfrac{K_2(b-a)^3}{24n^2}$.

- $\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$

- $\sin^2 x + \cos^2 x = 1$.
- $\tan^2 x + 1 = \sec^2 x$.
- $\frac{d}{dx}(\tan x) = \sec^2 x$.
- $\frac{d}{dx}(\sec x) = \sec x \tan x$.

- $\displaystyle\int \sin^n(x)\,dx = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$, for $n > 0$.
- $\displaystyle\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$, for $n > 0$.
- $\displaystyle\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$, for $n \neq 1$.
- $\displaystyle\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$, for $n \neq 1$.
- $\displaystyle\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$.

- $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.
- $|f(x) - P_n(x)| \leq \dfrac{K_{n+1}}{(n+1)!}|x-x_0|^{n+1}$.

- $\sin x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ on $(-\infty, \infty)$.
- $\cos x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ on $(-\infty, \infty)$.
- $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{x!} = 1 + x + \frac{x^2}{2!} + \ldots$ on $(-\infty, \infty)$.
- $\arctan x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ on $[-1, 1]$.