NAME:

Math 106B - Final Exam, April 11, 2007

INSTRUCTIONS: Show all of your work and justify your solutions. Circle your final answers. Cross out any unnecessary work. The last page of this exam contains formulæ which may or may not be helpful in various problems.

1. (7 points) Consider the integral $I = \int_0^3 e^{-x^2}\,dx$. Using a right-endpoint sum to approximate I, how many intervals are necessary to guarantee accuracy to within ± 0.005?

2. (8 points) Solve the initial value problem $y' = \dfrac{1+y^2}{x}$, $y(e) = 0$.

3. (10 points) Evaluate $\displaystyle\int \frac{dx}{(4-x^2)^{3/2}}$. (Hint: Use a trig substitution.)

4. (10 points) A conical tank (think ice cream cone with its point at the bottom) has height 10 feet. At the top of the conical tank, the radius is 7 feet. The tank contains water that is 6 feet deep at its deepest point. Set up an integral to calculate the amount of work to pump the water out of the tank to a spigot that is 2 feet above the top of the tank. (Water weighs 62.4 pounds per cubic foot.)

5. (12 points) Let $f(x) = \sqrt{x}$.

(a) Find $P_2(x)$, the degree-2 Taylor polynomial of $f(x)$ centered at $x = 1$.

(b) Use $P_2(x)$ to approximate $\sqrt{2}$. (Write your answer as a fraction instead of a decimal approximation.)

(c) According to Taylor's Theorem, what is the maximum approximation error committed by $P_2(x)$ on the interval $[1, 2]$?

(d) Are your answers to (b) and (c) consistent? Explain.

6. (8 points) Determine if the following series converges absolutely, converges conditionally, or diverges:

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}.$$

7. (12 points) Determine if the following series converge or diverge.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2+4}{100k^2+3k+5}.$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{5^k k!}{(2k)!}.$

8. (7 points) Find the first four non-zero terms of the power series centered at $x = 0$ for the function

$$f(x) = \frac{e^{2x}}{1 - x}.$$

9. (11 points) Find the interval of convergence for the power series

$$S(x) = \sum_{k=1}^{\infty} \frac{(x-3)^k}{k2^k}.$$

10. (15 points)

(a) Evaluate $\lim_{x\to 0} \frac{x - \frac{x^3}{3} - \arctan x}{x^5}$.

(b) Suppose $g(x) = \arctan(x)$.

i. Evaluate $g^{(99)}(0)$.

ii. Evaluate $g^{(100)}(0)$.

1	2	3	4	5	6	7	8	9	10	TOTAL

Potentially useful formulæ

- $|I - L_n| \le \dfrac{K_1(b-a)^2}{2n}$.
- $|I - R_n| \le \dfrac{K_1(b-a)^2}{2n}$.
- $|I - T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$.
- $|I - M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$.

- $\displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

- $\sin^2 x + \cos^2 x = 1$.
- $\tan^2 x + 1 = \sec^2 x$.
- $\frac{d}{dx}(\tan x) = \sec^2 x$.
- $\frac{d}{dx}(\sec x) = \sec x \tan x$.

- $\displaystyle\int \sin^n(x)\, dx = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx, \text{ for } n > 0.$
- $\displaystyle\int \cos^n(x)\, dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\, dx, \text{ for } n > 0.$
- $\displaystyle\int \tan^n(x)\, dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\, dx, \text{ for } n \neq 1.$
- $\displaystyle\int \sec^n(x)\, dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\, dx, \text{ for } n \neq 1.$
- $\displaystyle\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C.$

- $\displaystyle P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$
- $\displaystyle |f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}.$

- $\displaystyle \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \text{ on } (-\infty, \infty).$
- $\displaystyle \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \text{ on } (-\infty, \infty).$
- $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{x!} = 1 + x + \frac{x^2}{2!} + \dots \text{ on } (-\infty, \infty).$
- $\displaystyle \arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \text{ on } [-1, 1].$