1. Compute the radius and interval (including endpoints) of convergence for \( \sum_{n=1}^{\infty} \frac{(x + 3)^n}{(5^n)(2n)} \).

\[
\lim_{n \to \infty} \left| \frac{(x+3)^{n+1}}{5^{n+1} \cdot 2^{n+2}} \cdot \frac{5^n}{(x+3)^n} \cdot \frac{2^n}{2 \cdot 2^n} \right| = \left| \frac{x+3}{5} \right| < 1 \Rightarrow -1 < \frac{x+3}{5} < 1
\]

\[-5 < x + 3 < 5 \]

Now check endpoints.

\( x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{5^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{, which diverges (p = 1).} \)

\( x = -8 \Rightarrow \sum_{n=1}^{\infty} \frac{(-8+3)^n}{5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \text{, which converges by A.S.T.} \)

So, interval is \(-8 \leq x \leq 2\) and radius is 5.

2. Find the complete Taylor series (in summation notation) for \( f(x) = \ln(1 - x) \) about \( x = 0 \).

\[
f(0) = 0 \quad f'(0) = -1 \quad f''(0) = -1 \quad f'''(0) = -2 \quad f^{(4)}(0) = -6
\]

Series is \( 0 + (-1) x + \frac{(-1) x^2}{2} + \frac{(-1) x^3}{3!} + \frac{(-1) x^4}{4!} + \cdots \)

\[= -x - \frac{x^2}{2} - \frac{x^3}{3!} - \frac{x^4}{4} - \cdots \]

\[
= \sum_{n=1}^{\infty} \frac{-x^n}{n}
\]

3. Evaluate the following exactly.

(a) \( 1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + \cdots = (e^x \text{ with } x = -1) = \frac{e^{-1}}{1 - \frac{1}{1}} = \frac{1}{2} \)

(b) \( 8/3 - 8/9 + 8/27 - 8/81 + \cdots \text{ (geometric with } a = \frac{8}{9} \text{ and } r = -\frac{1}{3}) = \frac{\frac{8}{3}}{1 - \left(-\frac{1}{3}\right)} = \frac{8}{3} \cdot \frac{3}{2} = 2 \)

(c) \( \pi - \pi^3/6 + \pi^5/120 - \cdots \text{ (sin x with } x = \pi) \) \( = \sin \pi = 0 \)

4. Let \( f(x) = x^3 \sin(-5x^2) \).

(a) Write out the first 3 non-zero terms in the Taylor series about \( x = 0 \) for \( f(x) \).

\[
f(x) = x^3 \left( -5x^2 + \frac{(-5x^5)^3}{3!} + \frac{(-5x^5)^5}{5!} - \cdots \right)
\]

\[
= -5x^5 + \frac{125x^9}{3!} - \frac{3125x^{13}}{5!} + \cdots
\]

(b) Write out the complete series for \( f(x) \) in summation notation.

\[
f(x) = \sum_{n=1}^{\infty} \frac{x^{4n+1} \cdot 5^{2n-1} \cdot (-1)^n}{(2n-1)!}
\]
(c) Compute \( f^{(13)}(0) \). We know the coefficient on \( x^{13} \) is \( \frac{f^{(13)}(0)}{13!} \).

So, from (a), we see \( -3125 = \frac{f^{(13)}(0)}{13!} \),
or \( f^{(13)}(0) = \frac{-13! \cdot 3125}{13!} \).

(d) Compute \( \lim_{x \to 0} \frac{5x^5 + 5x^9}{\sqrt{x}} \).

\[
\lim_{x \to 0} \frac{5x^5 - 3125x^9}{5x^9} = \lim_{x \to 0} \left( \frac{125}{31.5} - \frac{3125x^4}{51.5} + \ldots \right)
\]

\[
\frac{125}{31.5} = \frac{25}{6}
\]

5. Find the general solution of the differential equation \( \frac{dy}{dx} (1 + x^3) = x^2 e^y \).

\[
\int \frac{dy}{e^{2y}} = \int \frac{x^2 \, dx}{1 + x^3}
\]

\[
\int e^{-2y} \, dy = \int \frac{1}{3} \frac{dw}{w}
\]

\[
e^{-2y} = \frac{1}{3} \ln|w| + C
\]

\[
e^{-2y} = -\frac{1}{3} \ln|1 + x^3| + D
\]

How could you check that your solution is correct?

6. Sketch the slope field for \( \frac{dy}{dx} = y - 4x \).
7. Use Euler’s Method with 3 steps to estimate \( y(3/4) \) if \( dy/dx = y - 4x \) and \( y(0) = 2 \) and decide if your answer is too large or too small. 
\[
\Delta x = \frac{3/4 - 0}{3} = \frac{1}{4}
\]
\[
\begin{array}{c|c|c}
\text{X} & \text{Y} & \frac{dy}{dx} \cdot \Delta x = \Delta y \\
\hline
0 & 2 & 2 \cdot \frac{1}{4} = \frac{1}{2} \\
\frac{1}{4} & \frac{5}{2} & \frac{5}{2} \cdot \frac{1}{4} = \frac{5}{8} \\
\frac{1}{2} & \frac{23}{6} & \frac{23}{6} \cdot \frac{1}{4} = \frac{23}{32} \\
\frac{3}{4} & \frac{99}{32} & \\
\end{array}
\]

Because the solution through \((0, 2)\) is concave down (see slope field in 46), our estimate is too large.

8. A colony of endangered sea otters has an annual birth rate of 10% and an annual death rate of 15%. In an attempt to sustain the colony, activists bring in otters from another region where the animals are plentiful. They do this at a rate of 50 otters per year.

(a) Write a DE whose solution is \( P(t) \), the otter population \( t \) years from now.
\[
\frac{dp}{dt} = .1P - .15P + 50 = -0.05P + 50 \implies \frac{dp}{dt} = -0.05P + 50
\]

(b) Find any and all equilibrium solutions.
\[
\frac{dp}{dt} = -0.05 \left( P - (1000) \right) = 0 \implies P = 1000
\]

(c) Find the general solution of your DE.
\[
\int \frac{dp}{P-1000} = \int -0.05 \, dt
\]
\[
\ln|P-1000| = -0.05t + C
\]
\[
P - 1000 = \pm e^C e^{-0.05t}
\]
\[
P = (1000 + A) e^{-0.05t}
\]

(d) Find and sketch the particular solution if the current population is 400 otters.

\( P = 1000 \) is a stable equilib. soln.

9. A population obeys the differential equation \( \frac{dP}{dt} = 0.001P(3000 - P) \). Sketch solutions for \( P(t) \) for the following initial populations: \( P(0) = 0 \), \( P(0) = 100 \), \( P(0) = 2000 \), \( P(0) = 4000 \).
10. \( x(t) \) and \( y(t) \) give the populations (in 1000s) of two interacting groups of animals. Their interaction is described by the following system of differential equations.

\[
\frac{dx}{dt} = 2x - \frac{1}{2}xy \\
\frac{dy}{dt} = y(3 - y) + \frac{1}{6}xy
\]

(a) In the absence of population \( y \), how does population \( x \) behave? What happens to \( x \) as \( t \to \infty \)?

\[
\frac{dx}{dt} = 2x \quad \text{\( x \) grows exponentially.} \\
\lim_{t \to \infty} x(t) = \infty \quad \text{assuming \( x(0) \neq 0 \).}
\]

(b) In the absence of population \( x \), how does population \( y \) behave? What happens to \( y \) as \( t \to \infty \)?

\[
\frac{dy}{dt} = y(3 - y) \quad \text{\( y \) grows logistically.} \\
\lim_{t \to \infty} y(t) = 3 \quad \text{assuming \( y(0) \neq 0 \).}
\]

(c) Characterize the interaction between the two populations.

The presence of \( y \) hurts \( x \), but the presence of \( x \) helps \( y \), so it's likely that \( x \) is prey and \( y \) is predator.

(d) Find the nullclines of this system.

\[
\frac{dx}{dt} = x(2 - \frac{1}{2}y) = 0 \\
\frac{dy}{dt} = y(3 - y + \frac{1}{6}x) = 0
\]

\( \Rightarrow x = 0 \) or \( y = 4 \) \( \Rightarrow y = 0 \) or \( y = 3 + \frac{1}{6}x \).

(e) In the \( xy \)-plane, sketch the nullclines and indicate with an arrow the direction of the trajectories in each sector.

(f) Sketch the trajectory that starts at the point \((10,2)\).

See above.

See old exams and quizzes at http://abacus.bates.edu/~etowne/mathresources.html