Name:______________________________

Mathematics 205
Final Exam
April 10, 2012

Problem	Possible	Actual
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

You must show all work to receive credit.
No electronic devices other than calculators are permitted.
Give exact answers (such as $\ln 5$ or e^2) unless requested otherwise.

1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Are these vectors linearly independent? Explain.

   (b) Are these any of vectors orthogonal? Explain.

   (c) Let $W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Find an orthogonal basis for W.

2. Recall the difference between $\sqrt{4}$ and solving $x^2 = 4$. The former is 2 while the latter has the solutions $x = 2$ or $x = -2$. We call that positive root of $x^2 - 4 = 0$ the principal root of 4 and write $\sqrt{4}$. We can extend the definition to matrices. By diagonalizing a matrix and taking the square roots of the eigenvalues, we may compute the square root of a matrix.

   (a) Let $A = \begin{bmatrix} 9 & 15 \\ 0 & 4 \end{bmatrix}$. Compute $\sqrt{A}$.

   (b) Write all solutions to $x^2 = A$.

   (c) Suppose A was a 3×3 matrix with 3 positive, distinct eigenvalues. How many solutions would there be to $x^2 = A$?

3. We will use the following three statements in the Invertible Matrix Theorem.

   - A is an invertible matrix.
   - There is an $n \times n$ matrix C such that $CA = I$.
   - There is an $n \times n$ matrix D such that $AD = I$.

   (a) Show that if AB is invertible then A is invertible. You may not assume that B is invertible to do this problem.

   (b) Show that if AB is invertible then B is invertible. You may not assume that A is invertible to do this problem.

4. Let $\mathcal{K}$ be the set of 3×3 skew-symmetric matrices ($A = -A^T$) and let $\mathcal{S}$ be the set of 3×3 symmetric matrices ($A = A^T$). Let $T : \mathcal{K} \to \mathcal{S}$ be defined by $T(A) = A^2$.

   (a) Verify that the square of a skew-symmetric matrix is a symmetric matrix so that the statement "$T : \mathcal{K} \to \mathcal{S}$" makes sense. Recall that this is read as "T is a map from skew-symmetric matrices to symmetric matrices."

   (b) Is T a linear map?

   (c) What does it mean for a map to be onto? Is T onto? (Hint: think of the dimensions of the spaces involved.)

   (d) What does it mean for a map to be one-to-one? Is T one-to-one?

5. Suppose $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & 0 & k \\ 2 & 5 & 6 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$.

(a) If $k = 1$, what is $\det(B)$?

(b) What value of k makes B not invertible?

6. Balance the following chemical reaction using techniques learned in class.

$$PbN_6 + CrMn_2O_8 \rightarrow Pb_3O_4 + Cr_2O_3 + MnO_2 + NO.$$

7. The problem deals with the vector space of $n \times n$ matrices $\mathcal{M}_{n\times n}$.

(a) Explain why $\dim \mathcal{M}_{n\times n} = n^2$.

(b) Let $A \in \mathcal{M}_{n\times n}$. Show that there are scalars $c_0, c_1, c_2, \ldots, c_{n^2}$, not all 0, so that $c_0 I_n + c_1 A + c_2 A^2 + \ldots + c_{n^2} A^{n^2} = O$. That is, there is a nonzero polynomial p of degree at most n^2 so that $p(A) = O$ (where O is the $n \times n$ zero matrix).