

Name: \_\_\_\_\_

**Mathematics 205**  
**Final Exam**  
**April 10, 2012**

Problem	Possible	Actual
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

You must show all work to receive credit.

No electronic devices other than calculators are permitted.

Give exact answers (such as  $\ln 5$  or  $e^2$ ) unless requested otherwise.

1. Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ .

(a) Are these vectors linearly independent? Explain.

(b) Are these any of vectors orthogonal? Explain.

(c) Let  $W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ . Find an orthogonal basis for  $W$ .

2. Recall the difference between  $\sqrt{4}$  and solving  $x^2 = 4$ . The former is 2 while the latter has the solutions  $x = 2$  or  $x = -2$ . We call that positive root of  $x^2 - 4 = 0$  the principal root of 4 and write  $\sqrt{4}$ . We can extend the definition to matrices. By diagonalizing a matrix and taking the square roots of the eigenvalues, we may compute the square root of a matrix.

(a) Let  $A = \begin{bmatrix} 9 & 15 \\ 0 & 4 \end{bmatrix}$ . Compute  $\sqrt{A}$ .

(b) Write all solutions to  $x^2 = A$ .

(c) Suppose  $A$  was a  $3 \times 3$  matrix with 3 positive, distinct eigenvalues. How many solutions would there be to  $x^2 = A$ ?

3. We will use the following three statements in the Invertible Matrix Theorem.

- $A$  is an invertible matrix.
- There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .

(a) Show that if  $AB$  is invertible then  $A$  is invertible. You may not assume that  $B$  is invertible to do this problem.

(b) Show that if  $AB$  is invertible then  $B$  is invertible. You may not assume that  $A$  is invertible to do this problem.

4. Let  $\mathcal{K}$  be the set of  $3 \times 3$  skew-symmetric matrices ( $A = -A^T$ ) and let  $\mathcal{S}$  be the set of  $3 \times 3$  symmetric matrices ( $A = A^T$ ). Let  $T : \mathcal{K} \rightarrow \mathcal{S}$  be defined by  $T(A) = A^2$ .

(a) Verify that the square of a skew-symmetric matrix is a symmetric matrix so that the statement " $T : \mathcal{K} \rightarrow \mathcal{S}$ " makes sense. Recall that this is read as " $T$  is a map from skew-symmetric matrices to symmetric matrices."

(b) Is  $T$  a linear map?

(c) What does it mean for a map to be onto? Is  $T$  onto? (Hint: think of the dimensions of the spaces involved.)

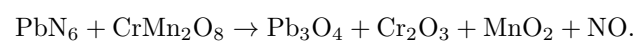
(d) What does it mean for a map to be one-to-one? Is  $T$  one-to-one?

5. Suppose  $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & 0 & k \\ 2 & 5 & 6 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$ .

(a) If  $k = 1$ , what is  $\det(B)$ ?

(b) What value of  $k$  makes  $B$  not invertible?

6. Balance the following chemical reaction using techniques learned in class.



7. The problem deals with the vector space of  $n \times n$  matrices  $\mathcal{M}_{n \times n}$ .

(a) Explain why  $\dim \mathcal{M}_{n \times n} = n^2$ .

(b) Let  $A \in \mathcal{M}_{n \times n}$ . Show that there are scalars  $c_0, c_1, c_2, \dots, c_{n^2}$ , not all 0, so that  $c_0 I_n + c_1 A + c_2 A^2 + \dots + c_{n^2} A^{n^2} = O$ . That is, there is a nonzero polynomial  $p$  of degree at most  $n^2$  so that  $p(A) = O$  (where  $O$  is the  $n \times n$  zero matrix).