1A. Let $A = \begin{bmatrix} -1 & 4 & -6 \\ 0 & 3 & 0 \\ -2 & 2 & 0 \end{bmatrix}$ and verify that $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ is an eigenvector of A.
What is the corresponding eigenvalue?

1B. Fact: Another eigenvalue of A is $\lambda = -4$. Find a basis for the eigenspace of $\lambda = -4$.

1C. Fact: There are only two distinct eigenvalues for this matrix (so one of them has multiplicity 2). Show that A *is* diagonalizable; exhibit appropriate P and D that demonstrate this.

1D. What is the characteristic polynomial of A, in factored form?

2A. Fact: The matrix $B = \begin{bmatrix} -22 & 25 & 36 \\ -1 & 4 & 2 \\ -12 & 12 & 20 \end{bmatrix}$ has exactly the same characteristic polynomial as matrix A in problem 1, yet B is *not* diagonalizable. What must "go wrong"?

2B. Support your conclusion: your answer will be in terms of one of the eigenspaces of B.

3. Let $C = \begin{bmatrix} 25 & 58 & 11 & 36 \\ 1 & 4 & 1 & 2 \\ 12 & 27 & 5 & 17 \end{bmatrix}$, and let R be the Reduced Row Echelon Form of C.

Find a basis for each of the following: (you may write "same as" if the basis is the same as the answer to an earlier one — eg, "same as 3A"). Various helpful row reductions appear elsewhere on this exam.

3A. $\mathrm{Col}(C)$

3B. $\mathrm{Col}(C^T)$

3C. $\mathrm{Null}(C)$

3D. $\mathrm{Null}(C^T)$

3E. $\mathrm{Row}(C)$

3F. $\mathrm{Row}(R)$

3G. $\mathrm{Null}(R)$

3H. $(\mathrm{Null}(C))^\perp$

3I. $(\mathrm{Col}(C))^\perp$

4A. In terms of $\mathbf{x}$ (the amount produced), C (the consumption matrix) and $\mathbf{d}$ (the final demand), what is the equation used to model the Leontief Input-Output Model for an economy?

4B. Suppose a certain economy has three producing sectors, Math, Music and Food, and an open sector of People who just consume Math, Music and Food while producing nothing. As it turns out, the silly People demand no Math, but lots of Food, 708 units worth, and an amount of Music, S, which we will determine by the end of the problem. Making one unit of Math consumes 4/10 units of Math, 0.15 of one unit of Music, and an as yet unknown amount f of a unit of Food, whereas Music requires 1/10 of a unit *each* of Music and Food, but an unknown amount m of a unit of Math, and Food requires 0.05, 0.25 and 0.2 units of Math, Music and Food, respectively. The production levels of Math, Music and Food are 360, 800, and 1120 units.

Explicitly write out the Leontief Input-Output Model for this system in matrix notation. There will be three unkowns: f, m and S and lots of numbers in the matrix equation you write.

4C. Find f, m and S.

4D. Explicitly find the intermediate demands for Math, Music, and Food.

5. Suppose T is a linear transformation with standard matrix $C = \begin{bmatrix} 25 & 58 & 11 & 36 \\ 1 & 4 & 1 & 2 \\ 12 & 27 & 5 & 17 \end{bmatrix}$ (the same C as in problem 3).

5A. Find any conditions on $\mathbf{b} = (b_1, b_2, b_3)$ (written sideways to save space) such that $\mathbf{b}$ is in the image of T. Explain your answer.

5B. Give an explicit example of a vector not in the image of T.

5C. Find $T(\mathbf{w})$ where (written sideways) $\mathbf{w} = (2, 0, 0, 1)$.

5D. Recall the kernel of a linear transformation T is the set of all vectors in the domain which T maps to the zero-vector. Find a basis for this subspace of T.

6A. A "catenary" is the name of the curve traced out by a flexible chain suspended at its ends and hanging freely. Galileo declared such a curve is really a parabola, but in fact this is incorrect. Just for fun, I hung one of my wife's necklaces over my computer monitor, which was displaying a coordinate system; a photograph appears below. Sure looks parabolic! The four data points I observed on the curve are at (0.6, 2), (5, −0.5), (6.3, 0) and (8, 2). Determine if they *do* lie on a parabola of the form $y = \beta_0 + \beta_1 x + \beta_2 x^2$ by setting up the appropriate matrix equation and row reducing it. Show all your results to two decimal places. (Hint: one of the four linear equations represented by your matrix equation will be $\beta_0 + \beta_1 \cdot 8 + \beta_2 \cdot 64 = 2$.

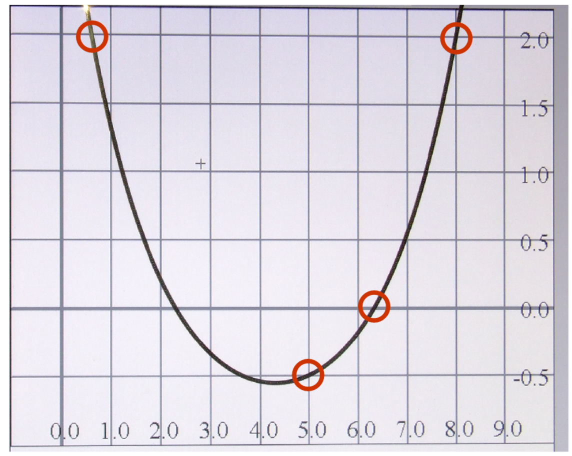

Fig. 1 Necklace Catenary

6B. Find the "best-fit" parabola using the method discussed in class. (Find β_0, β_1, and β_2).

6C. What is the projection of the four y-coordinates on the catenary onto the column space of the design matrix in this problem?

6D. What is are the y coordinates on your best-fit parabola?

6F. What is the "distance" in $\mathbf{R}^4$ between these two sets of y-coordinates?

7. Let S be a set of vectors in $\mathbf{R}^n$.

7A. Define what it means for S to be *linearly independent*.

7B. Define what it means for S to be *orthogonal*.

7C. Suppose S is an orthogonal set of nonzero vectors. Prove S is linearly independent.

8. What is the dimension of each of the following vector spaces?

8A. $\mathbf{P}^3$

8B. $\text{Col}(I_3)$

8C. $\mathbf{F}$, our space of continuous functions.

8D. $\{\mathbf{0}\}$

8E. The eigenspace of $\lambda = -4$ in problem 1.

9. Let $M = \begin{bmatrix} 5 & a & 0 \\ 0 & 0 & 2 \\ 0 & 1 & b \end{bmatrix}$.

Find the determinants of each of the following matrices and write your answers in the boxes.

M ☐

3M ☐

M^3 ☐

$\begin{bmatrix} 5 & a & 0 \\ 10 & a+a & 2 \\ 10 & 2a+1 & b \end{bmatrix}$ ☐

$\begin{bmatrix} 5 & a & a+b \\ 1 & 1 & 1 \\ 6 & a+1 & a+b+1 \end{bmatrix}$ ☐

$2M + 3I_3$ ☐

$M^{-1}M^T$ ☐

$$\left[\begin{array}{cccc|ccc} 25 & 58 & 11 & 36 & 1 & 0 & 0 \\ 1 & 4 & 1 & 2 & 0 & 1 & 0 \\ 12 & 27 & 5 & 17 & 0 & 0 & 1 \end{array}\right]$$

is row equivalent to

$$\left[\begin{array}{cccc|ccc} 1 & 0 & -1/3 & 2/3 & 0 & -\frac{9}{7} & \frac{4}{21} \\ 0 & 1 & 1/3 & 1/3 & 0 & 4/7 & -1/21 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|cccc} 25 & 1 & 12 & 1 & 0 & 0 & 0 \\ 58 & 4 & 27 & 0 & 1 & 0 & 0 \\ 11 & 1 & 5 & 0 & 0 & 1 & 0 \\ 36 & 2 & 17 & 0 & 0 & 0 & 1 \end{array}\right]$$

is row equivalent to $\left[\begin{array}{ccc|cccc} 1 & 0 & 1/2 & 0 & 0 & -1/7 & 1/14 \\ 0 & 1 & -1/2 & 0 & 0 & \frac{18}{7} & -\frac{11}{14} \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right]$