Math 106: Review for Final Exam, Part II - SOLUTIONS

1. **Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.**

   We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.

$$f(x) = x^{1/3} \qquad f(27) = 3$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \qquad f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}} \qquad f''(27) = -\frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187}$$

   Now plug in to the Taylor polynomial formula with $x_0 = 27$.

$$P_2(x) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2$$

   Finally, evaluate at $x = 28$.

$$\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797$$

2. **What is the largest possible error that could have occurred in your previous estimate?**

   We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

   In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

   $K_3 = $ max of $|f'''(x)|$ on $[27, 28]$ = max of $\left|\frac{10}{27x^{8/3}}\right|$ on $[27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}$

   Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{\frac{10}{177147}}{3!}|28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$.

3. **Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.**

   (a) $\displaystyle\int_1^\infty \frac{7 + 5\sin x}{x^2}\,dx$

   For all $x \geq 1$, we have $0 \leq \frac{7 + 5\sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = 12\frac{1}{x^2}$ because the maximum of $\sin x$ is 1.

$$\begin{aligned}12\int_1^\infty \frac{dx}{x^2} &= 12\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} \\ &= 12\lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t \\ &= 12\lim_{t\to\infty}\left[\frac{-1}{t} - \frac{-1}{1}\right] \\ &= 12[0 - (-1)] \\ &= 12\end{aligned}$$

   Therefore, the original integral in question must converge to a value less than 12.

   (b) $\displaystyle\int_1^\infty \frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}}\,dx$

   For $x \geq 1$, we have $\frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt[3]{10x^{12} + 17x^{12}}} \geq 0$. (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)

But $\dfrac{2x^3}{\sqrt[3]{10x^{12}+17x^{12}}} = \dfrac{2x^3}{\sqrt[3]{27x^{12}}} = \dfrac{2x^3}{3x^4} = \dfrac{2}{3}\dfrac{1}{x}$ and we know that $\dfrac{2}{3}\displaystyle\int_1^\infty \dfrac{dx}{x}$ diverges (compute for yourself or notice that $p = 1$).

Therefore the original integral must also diverge.

4. **Decide if each of the following sequences $\{a_k\}_{k=1}^{\infty}$ converges or diverges. If a sequence converges, compute its limit.**

   (a) $\boldsymbol{a_k = 3 + \dfrac{1}{10^k}}$ Terms are $3.1, 3.01, 3.001, 3.0001, \ldots$.

   $\displaystyle\lim_{k\to\infty}\left(3+\frac{1}{10^k}\right) = 3$, so the sequence converges to 3.

   (b) $\boldsymbol{a_k = (-1)^k}$ Terms are $-1, 1, -1, 1, \ldots$.

   $\displaystyle\lim_{k\to\infty}(-1)^k$ doesn't exist, so the sequence diverges.

   (c) $\boldsymbol{a_k = \dfrac{3+5k}{7+2k}}$ Terms are $8/9, 13/11, 18/13, 23/15, \ldots$.

   $\displaystyle\lim_{k\to\infty}\frac{3+5k}{7+2k} = \frac{5}{2}$ (by L'Hopital's Rule or by inspection), so the sequence converges to $\dfrac{5}{2}$.

5. **Decide if each of the following series converges or diverges. If a series converges, find its value.**

   (a) $\mathbf{3.1 + 3.01 + 3.001 + 3.0001 + \ldots}$

   $\displaystyle\lim_{k\to\infty} a_k = 3 \neq 0$, so the series diverges by the nth Term Test. (We keep adding 3's forever.)

   [Compare this with the first sequence of the previous problem.]

   (b) $\mathbf{1 + 1/2 + 1/3 + 1/4 + \ldots}$

   This is the famous Harmonic Series, which diverges *even though* the terms do approach 0. We can use the Integral Test: $\displaystyle\int_1^\infty \frac{dx}{x}$ diverges, which means that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ must diverge too.

   (c) $\mathbf{5 - 5/3 + 5/9 - 5/27 + \ldots}$

   This is a geometric series with $r = -\dfrac{1}{3}$, so it converges to $\dfrac{a}{1-r} = \dfrac{5}{1-(-1/3)} = \dfrac{15}{4}$.

6. **Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value. [The 1:10 section need not find the bounds.]**

   (a) $\displaystyle\boldsymbol{\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k+1}}}$ [Alternating Series Test]

   The terms of this series alternate in sign.

   And, $\dfrac{1}{\sqrt[3]{2}} \geq \dfrac{1}{\sqrt[3]{3}} \geq \dfrac{1}{\sqrt[3]{4}} \geq \ldots \geq 0$.

   And, $\displaystyle\lim_{k\to\infty}\frac{1}{\sqrt[3]{k+1}} = 0$.

   Therefore, by the Alternating Series Test, the series must converge.

   We know that any two consecutive partial sums will provide upper and lower bounds:

   lower bound $= S_1 = \dfrac{-1}{\sqrt[3]{2}}$ upper bound $= S_2 = \dfrac{-1}{\sqrt[3]{2}} + \dfrac{1}{\sqrt[3]{3}}$

   [To get better bounds, use later partial sums, such as S_5 and S_6.]

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{3^k (k!)^2}$ [Ratio Test]

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \frac{\frac{[2(k+1)]!}{3^{k+1}[(k+1)!]^2}}{\frac{(2k)!}{3^k(k!)^2}} \\
&= \lim_{k\to\infty} \frac{(2k+2)!}{(2k)!} \frac{3^k}{3^{k+1}} \frac{(k!)^2}{[(k+1)!]^2} \\
&= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{1} \frac{1}{3} \frac{1}{(k+1)^2} \\
&= \frac{4k^2+6k+2}{3k^2+6k+3} && \text{Use L'Hopital or divide each term by } k^2. \\
&= \frac{4}{3}
\end{aligned}$$

Since the limit of the ratio is greater than 1, the series diverges.

(c) $\displaystyle\sum_{k=1}^{\infty} (\frac{1}{100} + \frac{1}{k^5})$ [nth Term Test]

$\displaystyle\lim_{k\to\infty} (\frac{1}{100} + \frac{1}{k^5}) = \frac{1}{100} \neq 0$, so, by the nth Term Test, the series diverges.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{9k^8+5k^6}}{12k^5+3}$ [Comparison Test]

For $k \geq 1$, we have $\displaystyle\frac{\sqrt{9k^8+5k^6}}{12k^5+3} > \frac{\sqrt{9k^8}}{12k^5+3k^5} > 0$.

But, $\displaystyle\frac{\sqrt{9k^8}}{12k^5+3k^5} = \frac{3k^4}{15k^5} = \frac{1}{5}\frac{1}{k}$ and we know that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (use Integral Test or note that $p = 1$).

Therefore, the original series, which has larger terms, must diverge also.

(e) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$ [Integral Test]

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx \\
&= \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{u^2}\,du && \text{Substitute } u = \ln x, \text{so } du = \frac{dx}{x}. \\
&= \lim_{t\to\infty} \frac{u^{-1}}{-1}\Big|_{x=2}^{x=t} \\
&= \lim_{t\to\infty} \frac{-1}{\ln x}\Big|_2^t \\
&= \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \frac{-1}{\ln 2}\right] \\
&= 0 - \frac{-1}{\ln 2} \\
&= \frac{1}{\ln 2}
\end{aligned}$$

The integral converges, so the series must converge too.

Further, we know that $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx \le \sum_{k=2}^\infty \frac{1}{k(\ln(k))^2} \le a_2 + \int_2^\infty \frac{1}{x(\ln x)^2}\,dx.$

Therefore, our lower bound is $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln 2}.$

And our upper bound is $\displaystyle a_2 + \int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}.$

7. **Does the first series from the previous problem converge absolutely or conditionally?**

$\displaystyle\sum_{k=1}^\infty \left|\frac{(-1)^k}{\sqrt[3]{k+1}}\right| = \sum_{k=1}^\infty \frac{1}{\sqrt[3]{k+1}}$, which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges conditionally.

8. **Compute the radius and interval (including endpoints) of convergence for $\displaystyle\sum_{k=1}^\infty \frac{(x+3)^k}{k\cdot 5^k}$.**

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{\frac{(x+3)^{k+1}}{(k+1)\cdot 5^{k+1}}}{\frac{(x+3)^k}{k\cdot 5^k}}\right| &= \lim_{k\to\infty}\left|\frac{(x+3)^{k+1}}{(x+3)^k}\frac{k}{k+1}\frac{5^k}{5^{k+1}}\right| \\ &= \left|(x+3)\cdot 1\cdot\frac{1}{5}\right| \\ &= \left|\frac{x+3}{5}\right|\end{aligned}$$

Use L'Hopital on the middle fraction.

So, we are guaranteed convergence when $\left|\frac{x+3}{5}\right| < 1$. But this is equivalent to the following.

$$\begin{aligned}-1 < &\frac{x+3}{5} < 1 \\ -5 < &\, x+3 < 5 \\ -8 < &\quad x \quad < 2\end{aligned}$$

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At $x=2$, we have $\displaystyle\sum_{k=1}^\infty \frac{(2+3)^k}{k\cdot 5^k} = \sum_{k=1}^\infty \frac{5^k}{k\cdot 5^k} = \sum_{k=1}^\infty \frac{1}{k}$, which is the Harmonic Series and thus diverges.

At $x=-8$, we have $\displaystyle\sum_{k=1}^\infty \frac{(-8+3)^k}{k\cdot 5^k} = \sum_{k=1}^\infty \frac{(-5)^k}{k\cdot 5^k} = \sum_{k=1}^\infty \frac{(-1)^k}{k}$, which converges by the Alternating Series Test.

Thus, the interval of convergence is $-8 \le x < 2$ and the radius of convergence is 5.

9. **Find the complete Taylor series (in summation notation) for $f(x) = \ln(1-x)$ about $x=0$ and determine its interval of convergence.**

$$\begin{aligned} f(x) &= \ln(1-x) & f(0) &= 0 \\ f'(x) &= \frac{-1}{1-x} & f'(0) &= -1 \\ f''(x) &= \frac{-1}{(1-x)^2} & f'(0) &= -1 \\ f'''(x) &= \frac{-2}{(1-x)^3} & f'(0) &= -2 \\ f^{(4)}(x) &= \frac{-6}{(1-x)^4} & f'(0) &= -6 \end{aligned}$$

Now plug in to the Taylor series formula with $x_0 = 0$.

$$
\begin{aligned}
f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + ... &= 0 - 1(x) + \frac{-1}{2}x^2 + \frac{-2}{6}x^3 + ... \\
&= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ... \\
&= \sum_{k=1}^{\infty} \frac{-x^k}{k}
\end{aligned}
$$

We now find the interval of convergence as in the previous problem.

$$
\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{-x^{k+1}}{(k+1)}}{\frac{-x^k}{k}} \right| &= \lim_{k\to\infty} \left| \frac{-x^{k+1}}{-x^k} \frac{k}{k+1} \right| \qquad \text{Use L'Hopital on the second fraction.} \\
&= |x|
\end{aligned}
$$

So, we are guaranteed convergence for $|x| < 1$, which is equivalent to $-1 < x < 1$. Now check the endpoints.

At $x = 1$, we have $\sum_{k=1}^{\infty} \frac{-1^k}{k} = \sum_{k=1}^{\infty} \frac{-1}{k} = -\sum_{k=1}^{\infty} \frac{1}{k}$, which is the negative of the Harmonic Series and thus diverges.

At $x = -1$, we have $\sum_{k=1}^{\infty} \frac{-(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, which converges by the Alternating Series Test.

So, the interval of convergence is $-1 \le x < 1$.

10. (a) **Write the complete series equal to $\int_0^1 e^{-x^2}\,dx$ and show that it converges.**

We know $e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + ...$, so by substitution we obtain the following.

$$
e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + ... = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...
$$

$$
\begin{aligned}
\text{Thus, } \int_0^1 e^{-x^2}\,dx &= \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...\right] dx \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + ...\right]_0^1 \\
&= 1 - \frac{1^3}{3} + \frac{1^5}{5\cdot 2!} - \frac{1^7}{7\cdot 3!} + ... \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}
\end{aligned}
$$

The terms of this series alternate in sign.

And, $1 \ge \frac{1^3}{3} \ge \frac{1^5}{5\cdot 2!} \ge \frac{1^7}{7\cdot 3!} \ge ... \ge 0$.

And, $\lim_{k\to\infty} \frac{1}{(2k+1)k!} = 0$.

Therefore, by the Alternating Series Test, the series must converge.

(b) **If $f(x) = e^{-x^2}$, what is $f^{(400)}(0)$? What is $f^{(401)}(0)$?**

In the previous part, we found the following.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + ... = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$$

We know that $f^{(400)}(0)$ will appear in the Taylor series for $f(x)$ in the coefficient of the x^{400} term, which is $\frac{f^{(400)}(0)}{400!}x^{400}$.

From above, we see that term is $\frac{x^{400}}{200!}$.

Setting them equal gives $\frac{f^{(400)}(0)}{400!}x^{400} = \frac{x^{400}}{200!}$, which means that $f^{(400)}(0) = \frac{400!}{200!}$.

[Note that this does *not* simplify to 200!. Rather, $\frac{400!}{200!} = 400 \cdot 399 \cdot \ldots \cdot 202 \cdot 201$.]

We know that $f^{(401)}(0)$ will appear in the Taylor series for $f(x)$ in the coefficient of the x^{401} term, which is $\frac{f^{(401)}(0)}{401!}x^{401}$.

However, the Taylor series above for $f(x)$ has no terms with odd powers of x, meaning that the coefficients of all those terms must be 0.

Therefore, we know that $f^{(401)}(0) = 0$.