

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR AND THE AGREED-UPON PAGE OF NOTES, BUT NO BOOKS OR OTHER STUDENTS. GOOD LUCK!

1.) (10 pts.) Given the quadratic form $8x_1^2 + 6x_1x_2$,

a.) (2 pts.) find the symmetric matrix of the quadratic form;

$$A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}$$

b.) (2 pts.) classify the quadratic form as positive definite, negative definite, or indefinite, and explain your reasoning;

$$|A - \lambda I| = (8 - \lambda)(-\lambda) - 9 = \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1) = 0$$

when $\lambda = -1, 9$ so indefinite (eigenvalues \oplus and \ominus)

c.) (6 pts.) make a change of variable, $x = Py$, that transforms the quadratic form into one with no cross-product term.

$$\lambda_1 = 9: \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \therefore \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \text{ length} = \frac{\sqrt{9+1}}{\sqrt{10}}, \text{ so } \vec{v}_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$\lambda_2 = -1: \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix} \therefore \vec{x} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \text{ length} = \frac{1}{\sqrt{10}}, \text{ so } \vec{v}_2 = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\text{So } A = PDP^{-1} \text{ with } P = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}, D = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix}$$

Quadratic form with y -coordinates is then

$$9y_1^2 - y_2^2$$

2.) (15 pts.)

- a.) (5 pts.) **True or False:** A least-squares solution of $Ax = b$ is a vector \hat{x} such that $\|b - Ax\| \leq \|b - A\hat{x}\|$ for all x in \mathbb{R}^n . If this is true, explain why. If it is false, correct the statement to make it true.

False: the inequality should go the other way

$$\| \vec{b} - A\hat{x} \| \leq \| \vec{b} - A\vec{x} \|$$

- b.) (10 pts.) While boiling a pot of water, you take the temperature every two minutes. This generates the data points $(0, 15)$, $(2, 37)$, $(4, 68)$, $(6, 89)$, where the first coordinate is time, in minutes, and the second coordinate is temperature, in degrees Celsius. Using linear algebra techniques, find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits these data points.

| X | y |
|---|----|
| 0 | 15 |
| 2 | 37 |
| 4 | 68 |
| 6 | 89 |

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 37 \\ 68 \\ 89 \end{bmatrix} \quad \text{is } X\vec{\beta} = \vec{y}$$

Solve $X^T X \vec{\beta} = X^T \vec{y}$ to get least-squares solution

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix}$$

$$X^T \vec{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 15 \\ 37 \\ 68 \\ 89 \end{bmatrix} = \begin{bmatrix} 209 \\ 880 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 12 & 209 \\ 12 & 56 & 880 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 14.3 \\ 0 & 1 & 12.65 \end{array} \right] \quad \text{so } \beta_0 = 14.3$$

$$\beta_1 = 12.65$$

$$\text{and } y = 14.3 + 12.65x$$

3.) (15 pts.)

- a.) (5 pts.) Is it true that $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$ for every pair of vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n ? If so, explain why; if not, explain why not. Yes!

The dot product is commutative, that is,

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \quad \text{So } \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} \text{ does equal } 0.$$

- b.) (5 pts.) Suppose both U and V are orthogonal matrices. Explain why UV is an orthogonal matrix. That is, explain why UV is invertible and why its inverse is $(UV)^T$.

U and V are orthogonal, so each has an inverse,
and $U^{-1} = U^T$ and $V^{-1} = V^T$.

Since U and V are invertible, then UV is invertible,

$$\begin{aligned} \text{with } (UV)^{-1} &= V^{-1}U^{-1} && \downarrow \text{(Because } U, V \text{ orthogonal)} \\ &= V^T U^T && \downarrow \text{(transpose rule)} \\ &= (UV)^T \end{aligned}$$

- c.) (5 pts.) The Orthogonal Decomposition Theorem gives a formula for computing $\hat{\mathbf{y}}$, the projection of a vector \mathbf{y} onto a subspace W of \mathbb{R}^n . Each term in that formula is itself a projection of \mathbf{y} onto a subspace of W . Explain why.

$$\hat{\mathbf{y}} = \boxed{\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1} + \dots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p$$

Each term, for example the boxed term, is the projection of \vec{y} onto the subspace spanned by \vec{u}_1 (or \vec{u}_2 or... or \vec{u}_p).

(Each of these subspaces is one-dimensional: a line.)

4.) (15 pts.)

a.) (5 pts.) Verify that $\mathbf{v} = (2, 1, -1, 2)$ is an eigenvector of A , given below. What is the corresponding eigenvalue of \mathbf{v} ?

$$A = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ -5 \\ 10 \end{bmatrix} = 5\vec{v}$$

So yes, \vec{v} is an eigen vector, with corresponding eigenvalue $\lambda = 5$.

b.) (5 pts.) Construct a 4×4 matrix with eigenvalues -3 , 2 , and 5 (with multiplicity 2). Your matrix should not be strictly diagonal - that is, there must be some nonzero entries in non-diagonal positions within the matrix.

$$\begin{bmatrix} -3 & 6 & 7 & -5 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} : \text{Upper and lower triangular matrices also have eigenvalues on the main diagonal}$$

c.) (5 pts.) Use the factorization $A = PDP^{-1}$ to compute A^k , where k represents an arbitrary positive integer.

$$A = \begin{bmatrix} 33 & -20 \\ 60 & -37 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = PDP^{-1}$$

$$A^k = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-7)^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3^k - 3(-7)^k & -2 \cdot 3^k + 2(-7)^k \\ 6 \cdot 3^k - 6(-7)^k & -3 \cdot 3^k + 4(-7)^k \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3^k & (-7)^k \\ 3 \cdot 3^k & 2(-7)^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

5.) (15 pts.)

a.) (5 pts.) What are the three properties of a subspace H of \mathbb{R}^n ?

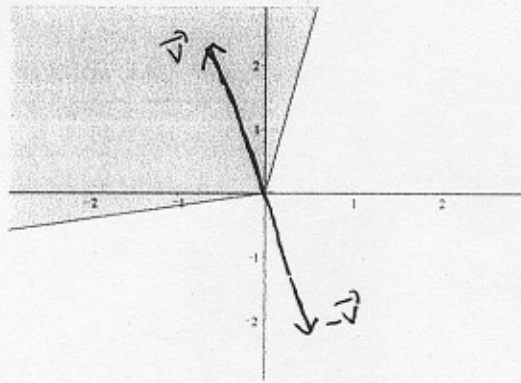
A subspace of \mathbb{R}^n is a set H in \mathbb{R}^n for which

1.) $\vec{0}$ is in H

2.) If \vec{u} and \vec{v} are in H , then $\vec{u} + \vec{v}$ is in H

3.) If \vec{u} is in H and c is a scalar,
then $c\vec{u}$ is in H .

b.) (5 pts.) The shaded region in the image below is a set in \mathbb{R}^2 . (Include the bounding lines as part of the set.) Give a specific reason why this set is *not* a subspace of \mathbb{R}^2 .



\vec{v} is in the set,
but $-\vec{v}$ (a scalar
multiple) is not,
so condition #3
(in part a.)
fails.

c.) (5 pts.) Let A be an $m \times n$ matrix. Explain why $\text{Nul } A$ is a subspace of \mathbb{R}^n .

1.) $\vec{0}$ is in $\text{Nul } A$: $A\vec{0} = \vec{0}$

2.) If \vec{u} and \vec{v} are in $\text{Nul } A$, then $A\vec{u} = A\vec{v} = \vec{0}$.

Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$, so $\vec{u} + \vec{v}$ is in
 $\text{Nul } A$

3.) If \vec{u} is in $\text{Nul } A$, then $A\vec{u} = \vec{0}$. Then for a
scalar c , $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$, so $c\vec{u}$ is
in $\text{Nul } A$.

6.) (15 pts.)

a.) (5 pts.) Must an elementary matrix be square? Why or why not?

Yes: it is an identity matrix (which is always square) after one row operation has been performed.

b.) (5 pts.) Let T be the linear transformation $T(x_1, x_2, x_3) = (4x_2 - 6x_3, 0, 7x_2 - 9x_3, x_1)$. Find the matrix A for which $T(\mathbf{x}) = A\mathbf{x}$.

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 4 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} -6 \\ 0 \\ -9 \\ 0 \end{bmatrix}$$

$$\text{So } A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)] = \begin{bmatrix} 0 & 4 & -6 \\ 0 & 0 & 0 \\ 0 & 7 & -9 \\ 1 & 0 & 0 \end{bmatrix}$$

c.) (5 pts.) Use the matrix inverse algorithm to compute A^{-1} , if it exists. If it does not exist, explain how the algorithm shows this.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

$$[A | I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array} \right]$$

row of zeroes

Left side is not row-equivalent to I , therefore A has no inverse.

7.) (15 pts.)

a.) (5 pts.) Suppose the vectors below are linearly independent. What can you say about the numbers $a, b, c, d, e,$ and f ?

None of $a, c,$ and f can be 0, or else $\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$ will not have 3 pivots.

$b, d,$ and e can be anything

b.) (5 pts.) Write the coefficient matrix of the system of equations below.

$$\begin{aligned} 3x_2 - 6x_3 + 8x_4 &= -5 \\ 3x_1 + x_3 - 2x_4 &= 7 \\ 4x_1 + x_2 + 5x_3 &= 8 \end{aligned}$$

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 3 & 0 & 1 & -2 \\ 4 & 1 & 5 & 0 \end{bmatrix}$$

c.) (5 pts.) Write the augmented matrix of the system of equations in part (b). Does the system have a solution? How do you know?

$$\left[\begin{array}{cccc|c} 0 & 3 & -6 & 8 & -5 \\ 3 & 0 & 1 & -2 & 7 \\ 4 & 1 & 5 & 0 & 8 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2/3 & 2.314 \\ 0 & 1 & 0 & 8/3 & -1.550 \\ 0 & 0 & 1 & 0 & 0.059 \end{array} \right]$$

There is a solution because there is a pivot in every row (Th. 4, Ch. 1)